Centre No.						Paper Reference	Surname	Initial(s)
Candidate No.						6 6 6 6 / 0 1	Signature	

Paper Reference(s)

6666/01

Edexcel GCE

Core Mathematics C4

Advanced

Monday 28 January 2013 – Morning

Time: 1 hour 30 minutes

Examiner's use only

Team Leader's use only

Question Number	Leave Blank
1	
2	
3	
4	
5	
6	
7	
8	
Total	

Materials required for examination	Items included with question papers
Mathematical Formulae (Pink)	Nil

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation or symbolic differentiation/integration, or have retrievable mathematical formulae stored in them.

Instructions to Candidates

In the boxes above, write your centre number, candidate number, your surname, initials and signature.
Check that you have the correct question paper.
Answer ALL the questions.
You must write your answer for each question in the space following the question.
When a calculator is used, the answer should be given to an appropriate degree of accuracy.

Information for Candidates

A booklet 'Mathematical Formulae and Statistical Tables' is provided.
Full marks may be obtained for answers to ALL questions.
The marks for individual questions and the parts of questions are shown in round brackets: e.g. **(2)**.
There are 8 questions in this question paper. The total mark for this paper is 75.
There are 28 pages in this question paper. Any blank pages are indicated.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You should show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.

This publication may be reproduced only in accordance with
Pearson Education Ltd copyright policy.
©2013 Pearson Education Ltd.

Printer's Log. No.
P41860A
W850/R6666/57570 5/5/5/6/

P 4 1 8 6 0 A 0 1 2 8

Turn over

PEARSON

1. Given

$$f(x) = (2 + 3x)^{-3}, \qquad |x| < \frac{2}{3}$$

find the binomial expansion of f(x), in ascending powers of x, up to and including the term in x^3.

Give each coefficient as a simplified fraction.

(5)

January 2013
6666 Core Mathematics C4
Mark Scheme

Question Number	Scheme		Marks
1.	$(2+3x)^{-3} = \underline{(2)^{-3}}\left(1+\frac{3x}{2}\right)^{-3} = \underline{\frac{1}{8}}\left(1+\frac{3x}{2}\right)^{-3}$	$\underline{(2)^{-3}}$ or $\frac{1}{8}$	B1
	$= \left\{\frac{1}{8}\right\}\left[1+(-3)(kx)+\frac{(-3)(-4)}{2!}(kx)^2+\frac{(-3)(-4)(-5)}{3!}(kx)^3+\dots\right]$	see notes	M1 A1
	$= \left\{\frac{1}{8}\right\}\left[\underline{1+(-3)\left(\frac{3x}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{3x}{2}\right)^3+\dots}\right]$		
	$= \frac{1}{8}\left[1-\frac{9}{2}x;+\frac{27}{2}x^2-\frac{135}{4}x^3+\dots\right]$	See notes below!	
	$= \frac{1}{8}-\frac{9}{16}x;+\frac{27}{16}x^2-\frac{135}{32}x^3+\dots$		A1; A1
			[5] **5**

B1: $\underline{(2)^{-3}}$ or $\underline{\frac{1}{8}}$ outside brackets or $\frac{1}{8}$ as constant term in the binomial expansion.

M1: Expands $(\dots+kx)^{-3}$ to give any 2 terms out of 4 terms simplified or un-simplified,

Eg: $1+(-3)(kx)$ or $(-3)(kx)+\frac{(-3)(-4)}{2!}(kx)^2$ or $1+\dots\dots+\frac{(-3)(-4)}{2!}(kx)^2$

or $\frac{(-3)(-4)}{2!}(kx)^2+\frac{(-3)(-4)(-5)}{3!}(kx)^3$ **where** $k \neq 1$ are ok for M1.

A1: A correct simplified or un-simplified $1+(-3)(kx)+\frac{(-3)(-4)}{2!}(kx)^2+\frac{(-3)(-4)(-5)}{3!}(kx)^3$ expansion with consistent (kx) where $k \neq 1$.

"Incorrect bracketing" $\left\{\frac{1}{8}\right\}\left[\underline{1+(-3)\left(\frac{3x}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{3x^2}{2}\right)+\frac{(-3)(-4)(-5)}{3!}\left(\frac{3x^3}{2}\right)+\dots}\right]$ is M1A0 unless recovered.

A1: For $\frac{1}{8}-\frac{9}{16}x$ **(simplified fractions)** or also allow $0.125-0.5625x$.

Allow Special Case A1 for either SC: $\frac{1}{8}\left[1-\frac{9}{2}x;\dots\right]$ or **SC:** $K\left[1-\frac{9}{2}x+\frac{27}{2}x^2-\frac{135}{4}x^3+\dots\right]$

(where K can be 1 or omitted), with each term in the $[\dots\dots\dots]$ either a simplified fraction or a decimal.

A1: Accept only $\frac{27}{16}x^2-\frac{135}{32}x^3$ or $1\frac{11}{16}x^2-4\frac{7}{32}x^3$ or $1.6875x^2-4.21875x^3$

1. ctd	Candidates who write $=\frac{1}{8}\left[\ 1+(-3)\left(-\frac{3x}{2}\right)+\frac{(-3)(-4)}{2!}\left(-\frac{3x}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(-\frac{3x}{2}\right)^3+\ldots\ \right]$ where $k=-\frac{3}{2}$ and not $\frac{3}{2}$ and achieve $\frac{1}{8}+\frac{9}{16}x+\frac{27}{16}x^2+\frac{135}{32}x^3+\ldots$ will get B1M1A1A0A0. ***<u>Alternative method:</u>*** Candidates can apply an alternative form of the binomial expansion. $(2+3x)^{-3} = (2)^{-3}+(-3)(2)^{-4}(3x)+\frac{(-3)(-4)}{2!}(2)^{-5}(3x)^2+\frac{(-3)(-4)(-5)}{3!}(2)^{-6}(3x)^3$ **B1:** $\frac{1}{8}$ or $(2)^{-3}$ **M1:** Any two of four (un-simplified) terms correct. **A1:** All four (un-simplified) terms correct. **A1:** $\frac{1}{8}-\frac{9}{16}x$ **A1:** $+\frac{27}{16}x^2-\frac{135}{32}x^3$ **Note:** The terms in C need to be evaluated, so ${}^{-3}C_0(2)^{-3}+{}^{-3}C_1(2)^{-4}(3x)+{}^{-3}C_2(2)^{-5}(3x)^2+{}^{-3}C_3(2)^{-6}(3x)^3$ without further working is B0M0A0.

Leave blank

2. (a) Use integration to find

$$\int \frac{1}{x^3} \ln x \, \mathrm{d}x$$

(5)

(b) Hence calculate

$$\int_1^2 \frac{1}{x^3} \ln x \, \mathrm{d}x$$

(2)

Question Number	Scheme	
2. (a)	$\int \frac{1}{x^3}\ln x\,\mathrm{d}x$, $\left\{\begin{array}{l} u=\ln x \;\Rightarrow\; \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x} \\ \frac{\mathrm{d}v}{\mathrm{d}x}=x^{-3} \;\Rightarrow\; v=\frac{x^{-2}}{-2}=\frac{-1}{2x^2}\end{array}\right\}$	
	In the form $\frac{\pm\lambda}{x^2}\ln x \pm \int \mu\,\frac{1}{x^2}.\frac{1}{x}$	M1
	$=\underline{\frac{-1}{2x^2}\ln x} - \underline{\underline{\int \frac{-1}{2x^2}.\frac{1}{x}\,\mathrm{d}x}}$ $\underline{\frac{-1}{2x^2}\ln x}$ simplified or un-simplified.	$\underline{\text{A1}}$
	$\underline{\underline{-\int \frac{-1}{2x^2}.\frac{1}{x}}}$ simplified or un-simplified.	$\underline{\underline{\text{A1}}}$
	$\left\{= \frac{-1}{2x^2}\ln x + \frac{1}{2}\int \frac{1}{x^3}\,\mathrm{d}x\right\}$	
	$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\left(-\frac{1}{2x^2}\right)\;\{+\,c\}$ $\pm\int \mu\,\frac{1}{x^2}.\frac{1}{x} \;\to\; \pm\beta x^{-2}.$	dM1
	Correct answer, with/without $+\,c$	A1
		[5]
(b)	$\left\{\left[-\frac{1}{2x^2}\ln x - \frac{1}{4x^2}\right]_1^2\right\} = \left(-\frac{1}{2(2)^2}\ln 2 - \frac{1}{4(2)^2}\right) - \left(-\frac{1}{2(1)^2}\ln 1 - \frac{1}{4(1)^2}\right)$ Applies limits of 2 and 1 to their part (a) answer and subtracts the correct way round.	M1
	$= \frac{3}{16} - \frac{1}{8}\ln 2$ or $\frac{3}{16} - \ln 2^{\frac{1}{8}}$ or $\frac{1}{16}(3 - 2\ln 2)$, etc, or awrt 0.1 or equivalent.	A1
		[2] **7**
(a)	**M1**: Integration by parts is applied in the form $\frac{\pm\lambda}{x^2}\ln x \pm \int \mu\,\frac{1}{x^2}.\frac{1}{x}$ or equivalent. $\underline{\textbf{A1}}$: $\underline{\frac{-1}{2x^2}\ln x}$ simplified or un-simplified. $\underline{\underline{\textbf{A1}}}$: $\underline{\underline{-\int \frac{-1}{2x^2}.\frac{1}{x}}}$ or equivalent. You can ignore the $\mathrm{d}x$. **dM1:** Depends on the previous M1. $\pm\int \mu\,\frac{1}{x^2}.\frac{1}{x} \;\to\; \pm\beta x^{-2}.$ **A1:** $-\frac{1}{2x^2}\ln x + \frac{1}{2}\left(-\frac{1}{2x^2}\right)\;\{+\,c\}$ or $= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2}\;\{+\,c\}$ or $\frac{x^{-2}}{-2}\ln x - \frac{x^{-2}}{4}\;\{+\,c\}$ or $\frac{-1-2\ln x}{4x^2}\;\{+\,c\}$ or equivalent. You can ignore subsequent working after a correct stated answer.	
(b)	**M1:** Some evidence of applying limits of 2 and 1 to their part (a) answer and subtracts the correct way round. **A1:** ***Two term exact answer*** of either $\frac{3}{16} - \frac{1}{8}\ln 2$ or $\frac{3}{16} - \ln 2^{\frac{1}{8}}$ or $\frac{1}{16}(3 - 2\ln 2)$ or $\frac{\ln\left(\frac{1}{4}\right) + 3}{16}$ or $0.1875 - 0.125\ln 2$. Also allow awrt 0.1. Also note the fraction terms must be combined. **Note:** Award the final A0 in part (b) for a candidate who achieves awrt 0.1 in part (b), when their answer to part (a) is incorrect.	

2. (b) ctd	**Note:** Decimal answer is 0.100856... in part (b).		
	Alternative Solution $\int \frac{1}{x^3}\ln x\,\mathrm{d}x\,,\quad \left\{\begin{array}{ll} u = x^{-3} \;\Rightarrow\; \frac{\mathrm{d}u}{\mathrm{d}x} = -3x^{-4} \\ \frac{\mathrm{d}v}{\mathrm{d}x} = \ln x \;\Rightarrow\; v = x\ln x - x \end{array}\right\}$		
	$\int \frac{1}{x^3}\ln x\,\mathrm{d}x = \frac{1}{x^3}(x\ln x - x) - \int (x\ln x - x)\frac{-3}{x^4}\mathrm{d}x$		
		$k\int \frac{1}{x^3}\ln x\,\mathrm{d}x = \frac{1}{x^3}(x\ln x - x) \pm \int \frac{\lambda}{x^3}\mathrm{d}x$ where $k \neq 1$	M1
	$-2\int \frac{1}{x^3}\ln x\,\mathrm{d}x = \frac{1}{x^3}(x\ln x - x) - \int \frac{3}{x^3}\mathrm{d}x$	Any one of $\frac{1}{x^3}(x\ln x - x)$ or $-\int \frac{3}{x^3}\mathrm{d}x$	A1
		$\frac{1}{x^3}(x\ln x - x) - \int \frac{3}{x^3}\mathrm{d}x$ and $k = -2$	A1
	$-2\int \frac{1}{x^3}\ln x\,\mathrm{d}x = \frac{1}{x^3}(x\ln x - x) + \frac{3}{2x^2}\ \{+c\}$	$\pm\int \mu\frac{1}{x^3} \rightarrow \pm\beta x^{-2}.$	dM1
	$\int \frac{1}{x^3}\ln x\,\mathrm{d}x = -\frac{1}{2x^3}(x\ln x - x) - \frac{3}{4x^2}\ \{+c\}$	$-\frac{1}{2x^3}(x\ln x - x) - \frac{3}{4x^2}$ or equivalent with/without $+c$.	A1
	$= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2}\ \{+c\}$		

Leave blank

3. Express $\dfrac{9x^2 + 20x - 10}{(x+2)(3x-1)}$ in partial fractions.

(4)

Question Number	Scheme		Marks
3.	**Method 1: Using one identity**		
	$\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv A + \frac{B}{(x+2)} + \frac{C}{(3x-1)}$		
	$A=3$	their constant term $= 3$	B1
	$9x^2+20x-10 \equiv A(x+2)(3x-1)+B(3x-1)+C(x+2)$	Forming a correct identity.	B1
	Either $x^2: \; 9=3A, \quad x: \; 20=5A+3B+C$ constant: $-10=-2A-B+2C$ or $x=-2 \Rightarrow 36-40-10=-7B \Rightarrow -14=-7B \Rightarrow B=2$	Attempts to find the value of either one of their *B* or their *C* from their identity.	M1
	$x=\frac{1}{3} \Rightarrow 1+\frac{20}{3}-10=\frac{7}{3}C \Rightarrow -\frac{7}{3}=\frac{7}{3}C \Rightarrow C=-1$	Correct values for their *B* and their *C*, which are found using a correct identity.	A1
			[4]
	Method 2: Long Division		
	$\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3 + \frac{5x-4}{(x+2)(3x-1)}$	their constant term $= 3$	B1
	So, $\frac{5x-4}{(x+2)(3x-1)} \equiv \frac{B}{(x+2)} + \frac{C}{(3x-1)}$		
	$5x-4 \equiv B(3x-1)+C(x+2)$	Forming a correct identity.	B1
	Either $x: \; 5=3B+C$, constant: $-4=-B+2C$ or $x=-2 \Rightarrow -10-4=-7B \Rightarrow -14=-7B \Rightarrow B=2$	Attempts to find the value of either one of their *B* or their *C* from their identity.	M1
	$x=\frac{1}{3} \Rightarrow \frac{5}{3}-4=\frac{7}{3}C \Rightarrow -\frac{7}{3}=\frac{7}{3}C \Rightarrow C=-1$	Correct values for their *B* and their *C*, which are found using $5x-4 \equiv B(3x-1)+C(x+2)$	A1
	So, $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3 + \frac{2}{(x+2)} - \frac{1}{(3x-1)}$		**[4]**
			4
	1st B1: Their constant term must be equal to 3 for this mark. **2nd B1 (M1 on epen):** Forming a correct identity. This can be implied by later working. **M1 (A1 on epen):** Attempts to find the value of either one of their *B* or their *C* from their identity. This can be achieved by ***either*** substituting values into their identity ***or*** comparing coefficients and solving the resulting equations simultaneously. **A1:** Correct values for their *B* and their *C*, which are found using a correct identity. **Note :** $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv \frac{A}{(x+2)} + \frac{B}{(3x-1)}$, leading to $9x^2+20x-10 \equiv A(3x-1)+B(x+2)$, leading to $A=2$ and $B=-1$ will gain a maximum of B0B0M1A0 .		

3. ctd

Note: You can imply the 2nd B1 from either $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv \frac{A(x+2)(3x-1)+B(3x-1)+C(x+2)}{(x+2)(3x-1)}$

or $\frac{5x-4}{(x+2)(3x-1)} \equiv \frac{B(3x-1)+C(x+2)}{(x+2)(3x-1)}$

Alternative Method 1: Initially dividing by (x + 2)

$$\frac{9x^2+20x-10}{"(x+2)"(3x-1)} \equiv \frac{9x+2}{(3x-1)} - \frac{14}{(x+2)(3x-1)}$$

$$\equiv 3+\frac{5}{(3x-1)} - \frac{14}{(x+2)(3x-1)}$$

B1: their constant term $= 3$

So, $\frac{-14}{(x+2)(3x-1)} \equiv \frac{B}{(x+2)} + \frac{C}{(3x-1)}$

$-14 \equiv B(3x-1)+C(x+2)$

B1: Forming a correct identity.

$\Rightarrow B = 2,\ C = -6$

M1: Attempts to find either one of their *B* or their *C* from their identity.

So, $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3+\frac{5}{(3x-1)} + \frac{2}{(x+2)} - \frac{6}{(3x-1)}$

and $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3 + \frac{2}{(x+2)} - \frac{1}{(3x-1)}$

A1: Correct answer in partial fractions.

Alternative Method 2: Initially dividing by (3x - 1)

$$\frac{9x^2+20x-10}{(x+2)"(3x-1)"} \equiv \frac{3x+\frac{23}{3}}{(x+2)} - \frac{\frac{7}{3}}{(x+2)(3x-1)}$$

$$\equiv 3+\frac{\frac{5}{3}}{(x+2)} - \frac{\frac{7}{3}}{(x+2)(3x-1)}$$

B1: their constant term $= 3$

So, $\frac{-\frac{7}{3}}{(x+2)(3x-1)} \equiv \frac{B}{(x+2)} + \frac{C}{(3x-1)}$

$-\frac{7}{3} \equiv B(3x-1)+C(x+2)$

B1: Forming a correct identity.

$\Rightarrow B = \frac{1}{3},\ C = -1$

M1: Attempts to find either one of their *B* or their *C* from their identity.

So, $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3+\frac{\frac{5}{3}}{(x+2)} + \frac{\frac{1}{3}}{(x+2)} - \frac{1}{(3x-1)}$

and $\frac{9x^2+20x-10}{(x+2)(3x-1)} \equiv 3 + \frac{2}{(x+2)} - \frac{1}{(3x-1)}$

A1: Correct answer in partial fractions.

4.

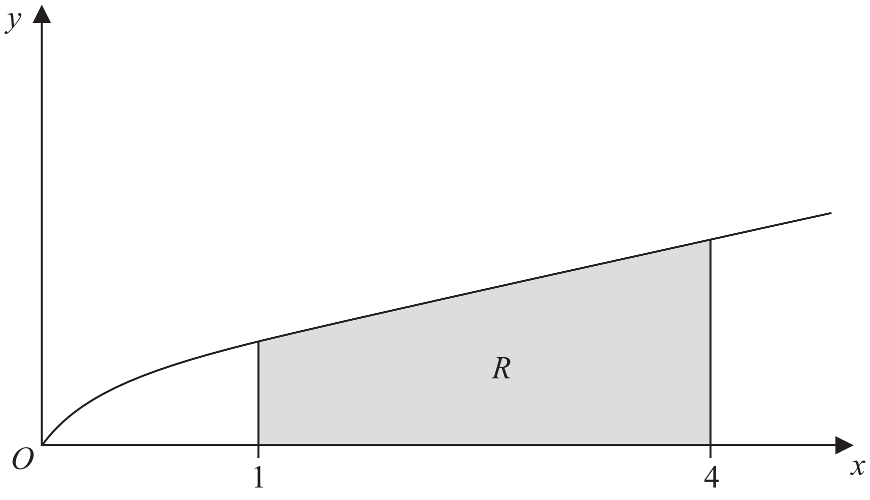

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{x}{1+\sqrt{x}}$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$.

(a) Complete the table with the value of y corresponding to $x = 3$, giving your answer to 4 decimal places.

(1)

x	1	2	3	4
y	0.5	0.8284		1.3333

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate of the area of the region R, giving your answer to 3 decimal places.

(3)

(c) Use the substitution $u = 1 + \sqrt{x}$, to find, by integrating, the exact area of R.

(8)

Question Number	Scheme	Marks
4. (a)	1.0981	B1 **cao** **[1]**
(b)	Area $\approx \frac{1}{2}\times 1\,;\times\underline{\left[0.5+2\left(0.8284+\text{their }1.0981\right)+1.3333\right]}$	B1; <u>M1</u>
	$= \frac{1}{2}\times 5.6863 \;=\; 2.84315 \;=\; 2.843\;\text{(3 dp)}$ 2.843 or awrt 2.843	A1 **[3]**
(c)	$\left\{u=1+\sqrt{x}\right\} \Rightarrow \underline{\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}x^{-\frac{1}{2}}}$ or $\underline{\frac{\mathrm{d}x}{\mathrm{d}u}=2(u-1)}$	<u>B1</u>
	$\int\frac{(u-1)^2}{u}......$	M1
	$\left\{\int\frac{x}{1+\sqrt{x}}\,\mathrm{d}x=\right\}\int\frac{(u-1)^2}{u}.\,2(u-1)\;\mathrm{d}u$ $\int\frac{(u-1)^2}{u}.\,2(u-1)$	A1
	$=\;2\int\frac{(u-1)^3}{u}\,\mathrm{d}u \;=\{2\}\int\frac{(u^3-3u^2+3u-1)}{u}\,\mathrm{d}u$ Expands to give a “four term” cubic in *u*. Eg: $\pm Au^3\pm Bu^2\pm Cu\pm D$	M1
	$=\;\{2\}\int\left(u^2-3u+3-\frac{1}{u}\right)\mathrm{d}u$ An attempt to divide at least three terms in ***their cubic*** by *u*. See notes.	M1
	$=\;\{2\}\left(\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right)$ $\int\frac{(u-1)^3}{u}\rightarrow\left(\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right)$	A1
	$\text{Area}(R)\;=\;\left[\frac{2u^3}{3}-3u^2+6u-2\ln u\right]_2^3$	
	$=\left(\frac{2(3)^3}{3}-3(3)^2+6(3)-2\ln 3\right)-\left(\frac{2(2)^3}{3}-3(2)^2+6(2)-2\ln 2\right)$ Applies limits of 3 and 2 in *u* or 4 and 1 in *x* and subtracts either way round.	M1
	$=\frac{11}{3}+2\ln 2-2\ln 3$ or $\frac{11}{3}+2\ln\left(\frac{2}{3}\right)$ or $\frac{11}{3}-\ln\left(\frac{9}{4}\right)$, etc Correct exact answer or equivalent.	A1 **[8]** **12**
(a)	**B1:** 1.0981 correct answer only. Look for this on the table or in the candidate’s working.	
(b)	**B1**: Outside brackets $\frac{1}{2}\times 1$ or $\frac{1}{2}$ **M1:** <u>For structure of trapezium rule</u>$\left[\;\underline{............}\;\right]$ **A1:** anything that rounds to 2.843 <u>**Note:**</u> Working must be seen to demonstrate the use of the trapezium rule. <u>**Note**</u>: actual area is 2.85573645… <u>Note:</u> Award B1M1 A1 for $\frac{1}{2}(0.5+1.3333)+\left(0.8284+\text{their }1.0981\right)=2.84315$ <u>Bracketing mistake</u>: **Unless the final answer implies that the calculation has been done correctly** Award B1M0A0 for $\frac{1}{2}\times 1\;+0.5+2\left(0.8284+\text{their }1.0981\right)+1.3333$ (nb: answer of 6.1863). Award B1M0A0 for $\frac{1}{2}\times 1\;(0.5+1.3333)+2\left(0.8284+\text{their }1.0981\right)$ (nb: answer of 4.76965).	

4. (b) ctd

Alternative method for part (b): Adding individual trapezia

$$\text{Area} \approx 1\times\left[\frac{0.5+0.8284}{2}+\frac{0.8284+1.0981}{2}+\frac{1.0981+1.3333}{2}\right] = 2.84315$$

B1: 1 and a divisor of 2 on all terms inside brackets.

M1: First and last ordinates once and two of the middle ordinates twice inside brackets ignoring the 2.

A1: anything that rounds to 2.843

(c)

B1: $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}$ or $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$ or $2\sqrt{x}\,\mathrm{d}u = \mathrm{d}x$ or $\mathrm{d}x = 2(u-1)\mathrm{d}u$ or $\frac{\mathrm{d}x}{\mathrm{d}u} = 2(u-1)$ oe.

1st M1: $\frac{x}{1+\sqrt{x}}$ becoming $\frac{(u-1)^2}{u}$ (Ignore integral sign).

1st A1 (B1 on epen): $\frac{x}{1+\sqrt{x}}\,\mathrm{d}x$ becoming $\frac{(u-1)^2}{u}.\,2(u-1)\{\mathrm{d}u\}$ or $\frac{(u-1)^2}{u}.\,\frac{2}{(u-1)^{-1}}\{\mathrm{d}u\}$.

You can ignore the integral sign and the $\mathrm{d}u$.

2nd M1: Expands to give a "four term" cubic in u, $\pm Au^3 \pm Bu^2 \pm Cu \pm D$

where $A \neq 0,\ B \neq 0,\ C \neq 0$ and $D \neq 0$ The cubic does not need to be simplified for this mark.

3rd M1: An attempt to divide at least three terms in ***their cubic*** by u.

Ie. $\frac{(u^3 - 3u^2 + 3u - 1)}{u} \to u^2 - 3u + 3 - \frac{1}{u}$

2nd A1: $\int \frac{(u-1)^3}{u}\,\mathrm{d}u \to \left(\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right)$

4th M1: Some evidence of limits of 3 and 2 in u and subtracting either way round.

3rd A1: Exact answer of $\frac{11}{3} + 2\ln 2 - 2\ln 3$ or $\frac{11}{3} + 2\ln\left(\frac{2}{3}\right)$ or $\frac{11}{3} - \ln\left(\frac{9}{4}\right)$ or $2\left(\frac{11}{6} + \ln 2 - \ln 3\right)$

or $\frac{22}{6} + 2\ln\left(\frac{2}{3}\right)$, etc. **Note**: that fractions must be combined to give either $\frac{11}{3}$ or $\frac{22}{6}$ or $3\frac{2}{3}$

Alternative method for 2nd M1 and 3rd M1 mark

$$\{2\}\int \frac{(u-1)^2}{u}.\,(u-1)\,\mathrm{d}u = \{2\}\int \frac{(u^2-2u+1)}{u}.\,(u-1)\,\mathrm{d}u$$

$$= \{2\}\int \left(u - 2 + \frac{1}{u}\right).\,(u-1)\,\mathrm{d}u = \{2\}\int \left(u^2 - ...\right)\,\mathrm{d}u$$

An attempt to expand $(u-1)^2$, then divide the result by u and then go on to multiply by $(u-1)$. **2nd M1**

$$= \{2\}\int \left(u^2 - 2u + 1 - u + 2 - \frac{1}{u}\right)\,\mathrm{d}u$$

to give three out of four of $\pm Au^2, \pm Bu, \pm C$ or $\pm\frac{D}{u}$ **3rd M1**

$$= \{2\}\int \left(u^2 - 3u + 3 - \frac{1}{u}\right)\,\mathrm{d}u$$

4. (c) ctd

Final two marks in part (c): $u = 1 + \sqrt{x}$

$$\text{Area}(R) = \left[\frac{2\left(1+\sqrt{x}\right)^3}{3} - 3\left(1+\sqrt{x}\right)^2 + 6\left(1+\sqrt{x}\right) - 2\ln\left(1+\sqrt{x}\right)\right]_1^4$$

$$= \left(\frac{2\left(1+\sqrt{4}\right)^3}{3} - 3\left(1+\sqrt{4}\right)^2 + 6\left(1+\sqrt{4}\right) - 2\ln\left(1+\sqrt{4}\right)\right)$$

$$-\left(\frac{2\left(1+\sqrt{1}\right)^3}{3} - 3\left(1+\sqrt{1}\right)^2 + 6\left(1+\sqrt{1}\right) - 2\ln\left(1+\sqrt{1}\right)\right)$$

M1: Applies limits of 4 and 1 in x and subtracts either way round.

$$= \left(18 - 27 + 18 - 2\ln 3\right) - \left(\frac{16}{3} - 12 + 12 - 2\ln 2\right)$$

$$= \frac{11}{3} + 2\ln 2 - 2\ln 3 \quad \text{or} \quad \frac{11}{3} + 2\ln\left(\frac{2}{3}\right) \quad \text{or} \quad \frac{11}{3} - \ln\left(\frac{9}{4}\right), \text{ etc}$$

A1: Correct exact answer or equivalent.

Alternative method for the final 5 marks in part (b)

$$\int \frac{(u-1)^3}{u}\,\mathrm{d}u\,, \qquad \left\{\begin{array}{lll} "u" = u^{-1} & \Rightarrow & \dfrac{\mathrm{d}"u"}{\mathrm{d}x} = -u^{-2} \\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = (u-1)^3 & \Rightarrow & v = \dfrac{(u-1)^4}{4} \end{array}\right\}$$

$$= \frac{(u-1)^4}{4u} - -\frac{1}{4}\int \frac{(u-1)^4}{u^2}\,\mathrm{d}u$$

$$= \frac{(u-1)^4}{4u} + \frac{1}{4}\int \frac{u^4 - 4u^3 + 6u^2 - 4u + 1}{u^2}\,\mathrm{d}u$$

M1: Applies integration by parts and expands to give a five term quartic.

$$= \frac{(u-1)^4}{4u} + \frac{1}{4}\int u^2 - 4u + 6 - \frac{4}{u} + \frac{1}{u^2}\,\mathrm{d}u$$

M1: Dividing at least 4 terms.

$$= \frac{(u-1)^4}{4u} + \frac{1}{4}\left(\frac{u^3}{3} - 2u^2 + 6u - 4\ln u - \frac{1}{u}\right)$$

A1: Correct Integration.

$$\int_2^3 \frac{(u-1)^3}{u}\,\mathrm{d}u = \left[\frac{(u-1)^4}{4u} + \frac{u^3}{12} - \frac{u^2}{2} + \frac{3u}{2} - \ln u - \frac{1}{4u}\right]_2^3$$

$$= \left(\frac{16}{12} + \frac{27}{12} - \frac{9}{2} + \frac{9}{2} - \ln 3 - \frac{1}{12}\right) - \left(\frac{1}{8} + \frac{8}{12} - \frac{4}{2} + \frac{6}{2} - \ln 2 - \frac{1}{8}\right)$$

M1

$$= \left(7 - \ln 3\right) - \left(\frac{5}{3} - \ln 2\right)$$

$$= \frac{11}{6} + \ln\frac{2}{3}$$

$$\text{Area}(R) = 2\int_2^3 \frac{(u-1)^3}{u}\,\mathrm{d}u = 2\left(\frac{11}{6} + \ln\frac{2}{3}\right)$$

A1

5.

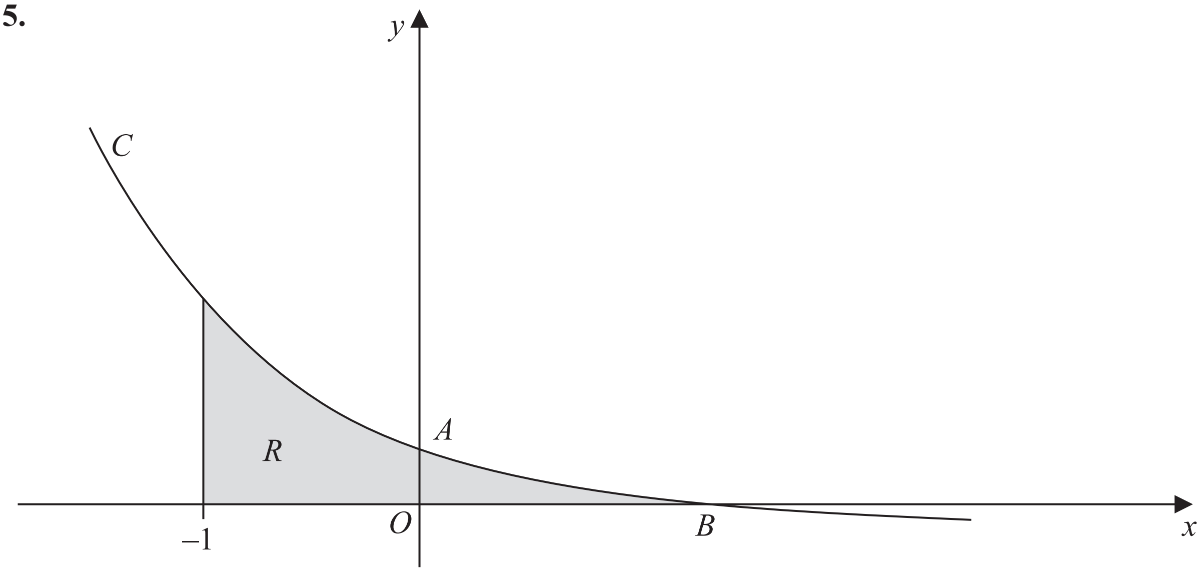

Figure 2

Figure 2 shows a sketch of part of the curve *C* with parametric equations

$$x = 1 - \frac{1}{2}t, \quad y = 2^t - 1$$

The curve crosses the *y*-axis at the point *A* and crosses the *x*-axis at the point *B*.

(a) Show that *A* has coordinates (0, 3).

(2)

(b) Find the *x* coordinate of the point *B*.

(2)

(c) Find an equation of the normal to *C* at the point *A*.

(5)

The region *R*, as shown shaded in Figure 2, is bounded by the curve *C*, the line $x = -1$ and the *x*-axis.

(d) Use integration to find the exact area of *R*.

(6)

Question Number	Scheme		Marks
5.	***Working parametrically:***		
	$x=1-\frac{1}{2}t,\quad y=2^t-1$ or $y=\mathrm{e}^{t\ln 2}-1$		
(a)	$\{x=0\Rightarrow\}\ 0=1-\frac{1}{2}t\Rightarrow t=2$	Applies $x=0$ to obtain a value for t.	M1
	When $t=2$, $y=2^2-1=3$	Correct value for y.	A1
			[2]
(b)	$\{y=0\Rightarrow\}\ 0=2^t-1\Rightarrow t=0$	Applies $y=0$ to obtain a value for t. (Must be seen in part (b)).	M1
	When $t=0$, $x=1-\frac{1}{2}(0)=1$	$x=1$	A1
			[2]
(c)	$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{2}$ and either $\frac{\mathrm{d}y}{\mathrm{d}t}=2^t\ln 2$ or $\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{t\ln 2}\ln 2$		B1
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2^t\ln 2}{-\frac{1}{2}}$	Attempts their $\frac{\mathrm{d}y}{\mathrm{d}t}$ divided by their $\frac{\mathrm{d}x}{\mathrm{d}t}$.	M1
	At A, $t="2"$, so $m(\mathbf{T})=-8\ln 2\Rightarrow m(\mathbf{N})=\frac{1}{8\ln 2}$	Applies $t="2"$ and $m(\mathbf{N})=\frac{-1}{m(\mathbf{T})}$	M1
	$y-3=\frac{1}{8\ln 2}(x-0)$ or $y=3+\frac{1}{8\ln 2}x$ or equivalent.	See notes.	M1 A1 oe cso
			[5]
(d)	$\mathrm{Area}(R)=\int\left(2^t-1\right).\left(-\frac{1}{2}\right)\mathrm{d}t$	Complete substitution for both y and $\mathrm{d}x$	M1
	$x=-1\rightarrow t=4$ and $x=1\rightarrow t=0$		B1
	$=\left\{-\frac{1}{2}\right\}\left(\frac{2^t}{\ln 2}-t\right)$	Either $2^t\rightarrow\frac{2^t}{\ln 2}$ or $\left(2^t-1\right)\rightarrow\frac{(2^t)}{\pm\alpha(\ln 2)}-t$ or $\left(2^t-1\right)\rightarrow\pm\alpha(\ln 2)(2^t)-t$	M1*
		$\left(2^t-1\right)\rightarrow\frac{2^t}{\ln 2}-t$	A1
	$\left\{-\frac{1}{2}\left[\frac{2^t}{\ln 2}-t\right]_4^0\right\}=-\frac{1}{2}\left(\left(\frac{1}{\ln 2}\right)-\left(\frac{16}{\ln 2}-4\right)\right)$	**Depends on the previous method mark.** Substitutes their changed limits in t and subtracts either way round.	dM1*
	$=\frac{15}{2\ln 2}-2$	$\frac{15}{2\ln 2}-2$ or equivalent.	A1
			[6] **15**

5. (a)	**M1:** Applies $x = 0$ and obtains a value of t. **A1:** For $y = 2^2 - 1 = 3$ or $y = 4 - 1 = 3$ ***Alternative Solution 1:*** **M1:** For substituting $t = 2$ into either x or y. **A1:** $x = 1 - \frac{1}{2}(2) = 0$ **and** $y = 2^2 - 1 = 3$ ***Alternative Solution 2:*** **M1:** Applies $y = 3$ and obtains a value of t. **A1:** For $x = 1 - \frac{1}{2}(2) = 0$ or $x = 1 - 1 = 0$. ***Alternative Solution 3:*** **M1:** Applies $y = 3$ or $x = 0$ and obtains a value of t. **A1:** Shows that $t = 2$ for both $y = 3$ and $x = 0$.
(b)	**M1:** Applies $y = 0$ and obtains a value of t. Working must be seen in part (b). **A1:** For finding $x = 1$. **Note:** Award M1A1 for $x = 1$.
(c)	**B1:** Both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ correct. This mark can be implied by later working. **M1:** Their $\frac{dy}{dt}$ divided by their $\frac{dx}{dt}$ or their $\frac{dy}{dt} \times \frac{1}{\text{their}\left(\frac{dx}{dt}\right)}$. **Note:** their $\frac{dy}{dt}$ must be a function of t. **M1:** Uses their value of t found in part (a) and applies $m(\mathbf{N}) = \frac{-1}{m(\mathbf{T})}$. **M1:** $y - 3 = (\text{their normal gradient})x$ **or** $y = (\text{their normal gradient})x + 3$ or equivalent. **A1:** $y - 3 = \frac{1}{8\ln 2}(x - 0)$ or $y = 3 + \frac{1}{8\ln 2}x$ or $y - 3 = \frac{1}{\ln 256}(x - 0)$ or $(8\ln 2)y - 24\ln 2 = x$ or $\frac{y - 3}{(x - 0)} = \frac{1}{8\ln 2}$. You can apply isw here. **Working in decimals** is ok for the three method marks. B1, A1 require exact values.
(d)	**M1:** Complete substitution for both y and dx. So candidate should write down $\int (2^t - 1).\left(\text{their } \frac{dx}{dt}\right)$ **B1:** Changes limits from $x \to t$. $x = -1 \to t = 4$ and $x = 1 \to t = 0$. Note $t = 4$ and $t = 0$ seen is B1. **M1*:** Integrates 2^t correctly to give $\frac{2^t}{\ln 2}$ ... or integrates $(2^t - 1)$ to give either $\frac{(2^t)}{\pm\alpha(\ln 2)} - t$ or $\pm\alpha(\ln 2)(2^t) - t$. **A1:** Correct integration of $(2^t - 1)$ with respect to t to give $\frac{2^t}{\ln 2} - t$. **dM1*: Depends upon the previous method mark.** Substitutes their limits in t and subtracts either way round. **A1:** Exact answer of $\frac{15}{2\ln 2} - 2$ or $\frac{15}{\ln 4} - 2$ or $\frac{15 - 4\ln 2}{2\ln 2}$ or $\frac{7.5}{\ln 2} - 2$ or $\frac{15}{2}\log_2 e - 2$ or equivalent.

Question Number	Scheme	Marks
5.	***Alternative: Converting to a Cartesian equation:*** $t = 2-2x \Rightarrow \quad y = 2^{2-2x} - 1$	
(a)	$\{x = 0 \Rightarrow\}\ y = 2^2 - 1$ Applies $x = 0$ in their Cartesian equation...	M1
	$y = 3$ ... to arrive at a correct answer of 3.	A1 **[2]**
(b)	$\{y = 0 \Rightarrow\}\ 0 = 2^{2-2x} - 1 \Rightarrow 0 = 2 - 2x \Rightarrow x = ...$ Applies $y = 0$ to obtain a value for x. (Must be seen in part (b)).	M1
	$x = 1$ $x = 1$	A1 **[2]**
(c)	$\frac{\mathrm{d}y}{\mathrm{d}x} = -2\left(2^{2-2x}\right)\ln 2$ $\pm\lambda 2^{2-2x},\ \lambda \neq 1$	M1
	$-2\left(2^{2-2x}\right)\ln 2$ or equivalent	A1
	At A, $x = 0$, so $m(\mathbf{T}) = -8\ln 2 \Rightarrow m(\mathbf{N}) = \frac{1}{8\ln 2}$ Applies $x = 0$ and $m(\mathbf{N}) = \frac{-1}{m(\mathbf{T})}$	M1
	$y - 3 = \frac{1}{8\ln 2}(x - 0)$ or $y = 3 + \frac{1}{8\ln 2}x$ or equivalent. ***As in the original scheme.***	M1 A1 oe **[5]**
(d)	$\text{Area}(R) = \int \left(2^{2-2x} - 1\right)\mathrm{d}x$ Form the integral of their Cartesian equation of C.	M1
	$= \int_{-1}^{1}\left(2^{2-2x} - 1\right)\mathrm{d}x$ For $2^{2-2x} - 1$ with limits of $x = -1$ and $x = 1$. Ie. $\int_{-1}^{1}\left(2^{2-2x} - 1\right)$	B1
	$= \left(\frac{2^{2-2x}}{-2\ln 2} - x\right)$ Either $2^{2-2x} \rightarrow \frac{2^{2-2x}}{-2\ln 2}$ or $\left(2^{2-2x} - 1\right) \rightarrow \frac{2^{2-2x}}{\pm\alpha(\ln 2)} - x$ or $\left(2^{2-2x} - 1\right) \rightarrow \pm\alpha(\ln 2)(2^{2-2x}) - x$	M1*
	$\left(2^{2-2x} - 1\right) \rightarrow \frac{2^{2-2x}}{-2\ln 2} - x$	A1
	$\left\{\left[\frac{2^{2-2x}}{-2\ln 2} - x\right]_{-1}^{1}\right\} = \left(\left(\frac{1}{-2\ln 2} - 1\right) - \left(\frac{16}{-2\ln 2} + 1\right)\right)$ **Depends on the previous method mark.** Substitutes limits of -1 and their x_B and subtracts either way round.	dM1*
	$= \frac{15}{2\ln 2} - 2$ $\frac{15}{2\ln 2} - 2$ or equivalent.	A1 **[6]** **15**
(d)	**Alternative method: In Cartesian and applying *u* = 2 – 2*x***	

	$\text{Area}(R) = \int \left(2^u - 1\right)\{\text{d}x\}$, where $u = 2 - 2x$ $= \int_4^0 \left(2^u - 1\right)\left(-\tfrac{1}{2}\right)\{\text{d}u\}$	**M0**: Unless a candidate ***writes*** $\int \left(2^{2-2x} - 1\right)\{\text{d}x\}$ Then apply the "working parametrically" mark scheme.

Question Number	Scheme		Marks
5. (d)	***Alternative method: For substitution*** $\mathbf{u = 2^t}$		
	$\text{Area}(R) = \int \left(2^t - 1\right).\left(-\frac{1}{2}\right)\text{d}t$	Complete substitution for both y and $\text{d}x$	M1
	where $u = 2^t \Rightarrow \frac{\text{d}u}{\text{d}t} = 2^t \ln 2 \Rightarrow \frac{\text{d}u}{\text{d}t} = u\ln 2$		
	$x = -1 \to t = 4 \to u = 16$ and $x = 1 \to t = 0 \to u = 1$	Both correct limits in t or both correct limits in u.	B1
	So $\text{area}(R) = -\frac{1}{2}\int \frac{u-1}{u\ln 2}\text{d}u$	If not awarded above, you can award M1 for this integral	
	$= -\frac{1}{2}\int \frac{1}{\ln 2} - \frac{1}{u\ln 2}\text{d}u$		
	$= \left\{-\frac{1}{2}\right\}\left(\frac{u}{\ln 2} - \frac{\ln u}{\ln 2}\right)$	Either $2^t \to \frac{u}{\ln 2}$ or $\left(2^t - 1\right) \to \frac{u}{\pm\alpha(\ln 2)} - \frac{\ln u}{\ln 2}$ or $\left(2^t - 1\right) \to \pm\alpha(\ln 2)(u) - \frac{\ln u}{\ln 2}$	M1*
		$\left(2^t - 1\right) \to \frac{u}{\ln 2} - \frac{\ln u}{\ln 2}$	A1
	$\left\{-\frac{1}{2}\left[\frac{u}{\ln 2} - \frac{\ln u}{\ln 2}\right]_{16}^{1}\right\} = -\frac{1}{2}\left(\left(\frac{1}{\ln 2}\right) - \left(\frac{16}{\ln 2} - \frac{\ln 16}{\ln 2}\right)\right)$	**Depends on the previous method mark.** Substitutes their changed limits ***in u*** and subtracts either way round.	dM1*
	$= \frac{15}{2\ln 2} - \frac{\ln 16}{2\ln 2}$ or $\frac{15}{2\ln 2} - 2$	$\frac{15}{2\ln 2} - \frac{\ln 16}{2\ln 2}$ or $\frac{15}{2\ln 2} - 2$ or equivalent.	A1
			[6]

Leave blank

6.

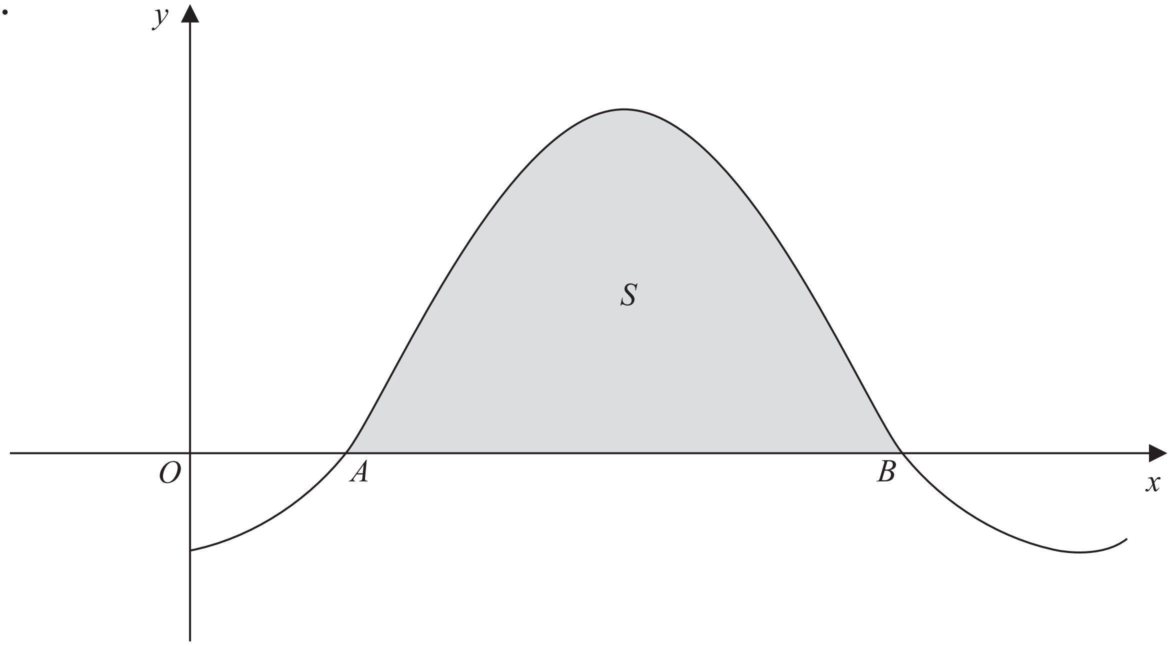

Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = 1 - 2\cos x$, where x is measured in radians. The curve crosses the x-axis at the point A and at the point B.

(a) Find, in terms of π, the x coordinate of the point A and the x coordinate of the point B.
(3)

The finite region S enclosed by the curve and the x-axis is shown shaded in Figure 3. The region S is rotated through 2π radians about the x-axis.

(b) Find, by integration, the exact value of the volume of the solid generated.
(6)

Question Number	Scheme		Marks
6. (a)	$\{y=0\Rightarrow\}\ 1-2\cos x=0$	$1-2\cos x=0$, seen or implied.	M1
	$\Rightarrow x=\frac{\pi}{3},\ \frac{5\pi}{3}$	At least one correct value of *x*. (See notes).	A1
		Both $\frac{\pi}{3}$ and $\frac{5\pi}{3}$	A1 **cso**
			[3]
(b)	$V=\pi\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}}(1-2\cos x)^2\ \mathrm{d}x$	For $\pi\int(1-2\cos x)^2$. Ignore limits and $\mathrm{d}x$	B1
	$\left\{\int(1-2\cos x)^2\ \mathrm{d}x\right\}=\int(1-4\cos x+4\cos^2 x)\mathrm{d}x$		
	$=\int 1-4\cos x+4\left(\frac{1+\cos 2x}{2}\right)\mathrm{d}x$	$\cos 2x=2\cos^2 x-1$ See notes.	M1
	$=\int\left(3-4\cos x+2\cos 2x\right)\mathrm{d}x$		
	$=3x-4\sin x+\frac{2\sin 2x}{2}$	Attempts $\int y^2$ to give any two of $\pm A\to\pm Ax$, $\pm B\cos x\to\pm B\sin x$ or $\pm\lambda\cos 2x\to\pm\mu\sin 2x$.	M1
		Correct integration.	A1
	$V=\{\pi\}\left(\left(3\left(\frac{5\pi}{3}\right)-4\sin\left(\frac{5\pi}{3}\right)+\frac{2\sin\left(\frac{10\pi}{3}\right)}{2}\right)-\left(3\left(\frac{\pi}{3}\right)-4\sin\left(\frac{\pi}{3}\right)+\frac{2\sin\left(\frac{2\pi}{3}\right)}{2}\right)\right)$	Applying limits the correct way round. Ignore π.	ddM1
	$=\pi\left(\left(5\pi+2\sqrt{3}-\frac{\sqrt{3}}{2}\right)-\left(\pi-2\sqrt{3}+\frac{\sqrt{3}}{2}\right)\right)$		
	$=\pi\left((18.3060...)-(0.5435...)\right)=17.7625\pi=55.80$		
	$=\pi\left(4\pi+3\sqrt{3}\right)$ or $4\pi^2+3\pi\sqrt{3}$	Two term exact answer.	A1
			[6] **9**

6. (a)	**M1:** $1-2\cos x=0$. This can be implied by either $\cos x=\frac{1}{2}$ or any one of the correct values for x in radians or in degrees. **1st A1:** Any one of either $\frac{\pi}{3}$ or $\frac{5\pi}{3}$ or 60 or 300 or awrt 1.05 or 5.23 or awrt 5.24 . **2nd A1:** Both $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.
(b)	**B1:** (M1 on epen) For $\pi\int(1-2\cos x)^2$. Ignore limits and $\mathrm{d}x$. **1st M1:** Any correct form of $\cos 2x=2\cos^2 x-1$ used or written down in the same variable. This can be implied by $\cos^2 x=\frac{1+\cos 2x}{2}$ or $4\cos^2 x \to 2+2\cos 2x$ or $\cos 2A=2\cos^2 A-1$. **2nd M1:** Attempts $\int y^2$ to give any two of $\pm A \to \pm Ax$, $\pm B\cos x \to \pm B\sin x$ or $\pm\lambda\cos 2x \to \pm\mu\sin 2x$. Do not worry about the signs when integrating $\cos x$ or $\cos 2x$ for this mark. Note: $\int(1-2\cos x)^2=\int 1+4\cos^2 x$ is ok for an attempt at $\int y^2$. **1st A1:** Correct integration. Eg. $3x-4\sin x+\frac{2\sin 2x}{2}$ or $x-4\sin x+\frac{2\sin 2x}{2}+2x$ oe. **3rd ddM1:** Depends on both of the two previous method marks. (Ignore π). Some evidence of substituting their $x=\frac{5\pi}{3}$ and their $x=\frac{\pi}{3}$ and subtracting the correct way round. You will need to use your calculator to check for correct substitution of their limits into their integrand if a candidate does not explicitly give **some evidence.** **Note:** For correct integral and limits decimals gives: $\pi\left((18.3060...)-(0.5435...)\right)=17.7625\pi=55.80$ **2nd A1:** ***Two term*** exact answer of either $\pi\left(4\pi+3\sqrt{3}\right)$ or $4\pi^2+3\pi\sqrt{3}$ or equivalent. **Note:** The π in the volume formula is only required for the B1 mark and the final A1 mark. **Note:** Decimal answer of 58.802... without correct exact answer is A0. **Note:** Applying $\int(1-2\cos x)\,\mathrm{d}x$ will usually be given no marks in this part.

7. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = (9\mathbf{i} + 13\mathbf{j} - 3\mathbf{k}) + \lambda(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

$$l_2 : \mathbf{r} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Given that l_1 and l_2 meet, find the position vector of their point of intersection.
(5)

(b) Find the acute angle between l_1 and l_2, giving your answer in degrees to 1 decimal place.
(3)

Given that the point A has position vector $4\mathbf{i} + 16\mathbf{j} - 3\mathbf{k}$ and that the point P lies on l_1 such that AP is perpendicular to l_1,

(c) find the exact coordinates of P.
(6)

Question Number	Scheme		Marks
7. (a)	$\mathbf{i}: \quad 9+\lambda = 2+2\mu \quad \mathbf{(1)}$ $\mathbf{j}: \quad 13+4\lambda = -1+\mu \quad \mathbf{(2)}$ $\mathbf{k}: \quad -3-2\lambda = 1+\mu \quad \mathbf{(3)}$	Any two equations. (Allow one slip).	M1
	Eg: $\mathbf{(2)}-\mathbf{(3)}: \ 16+6\lambda = -2$ or $\mathbf{(2)}-4\mathbf{(1)}: \ -23 = -9-7\mu$	An attempt to eliminate one of the parameters.	dM1
	Leading to $\lambda = -3$ or $\mu = 2$	Either $\lambda = -3$ or $\mu = 2$	A1
	$l_1: \mathbf{r} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}$ or $l_2: \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}$	See notes	ddM1 A1
			[5]
(b)	$\mathbf{d_1} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \ \mathbf{d}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$	Realisation that the dot product is required between $\pm A\mathbf{d_1}$ and $\pm B\mathbf{d}_2$.	M1
	$\cos\theta = \pm\left(\dfrac{2+4-2}{\sqrt{(1)^2+(4)^2+(-2)^2}\,.\sqrt{(2)^2+(1)^2+(1)^2}}\right)$	Correct **equation**.	A1
	$\cos\theta = \dfrac{4}{\sqrt{21}.\sqrt{6}} \Rightarrow \theta = 69.1238974... = 69.1 \text{ (1 dp)}$	awrt 69.1	A1
			[3]
(c)	$\overrightarrow{OA} = \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix}, \quad \overrightarrow{OP} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9+\lambda \\ 13+4\lambda \\ -3-2\lambda \end{pmatrix}$		
	$\overrightarrow{AP} = \begin{pmatrix} 9+\lambda \\ 13+4\lambda \\ -3-2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix} = \begin{pmatrix} \lambda+5 \\ 4\lambda-3 \\ -2\lambda \end{pmatrix}$		M1 A1
	$\overrightarrow{AP} \bullet \mathbf{d_1} = 0 \Rightarrow \begin{pmatrix} \lambda+5 \\ 4\lambda-3 \\ -2\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \lambda+5+16\lambda-12+4\lambda = 0$		dM1
	leading to $\{21\lambda - 7 = 0 \Rightarrow\} \ \lambda = \dfrac{1}{3}$	$\lambda = \dfrac{1}{3}$	A1
	Position vector $\overrightarrow{OP} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \dfrac{1}{3}\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9\frac{1}{3} \\ 14\frac{1}{3} \\ -3\frac{2}{3} \end{pmatrix}$ or $\begin{pmatrix} \frac{28}{3} \\ \frac{43}{3} \\ -\frac{11}{3} \end{pmatrix}$		ddM1 A1
			[6] **14**

7. (a)

M1: Writes down any two equations. Allow one slip.

dM1: Attempts to eliminate either λ or μ to form an equation in one parameter only.

A1: For either $\lambda = -3$ or $\mu = 2$. **Note**: candidates only need to find one of the parameters.

ddM1: For either substituting their value of λ into l_1 or their μ into l_2.

2$^{\text{nd}}$ A1: For either $\begin{pmatrix} 6 \\ 1 \\ 3 \end{pmatrix}$ or $6\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ or $\begin{pmatrix} 6 & 1 & 3 \end{pmatrix}$.

Note: Each of the method marks in this part are dependent upon the previous method marks.

(b)

M1: Realisation that the dot product is required between $\pm A\mathbf{d}_1$ and $\pm B\mathbf{d}_2$. Allow one slip in $\mathbf{d}_1 = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

A1: Correct application of the dot product formula $\mathbf{d}_1 \bullet \mathbf{d}_2 = \pm\,|\mathbf{d}_1||\mathbf{d}_2|\cos\theta$ or $\cos\theta = \pm\left(\frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}\right)$

The dot product must be correctly applied and the square roots although they can be un-simplified must be correctly applied.

A1: awrt 69.1 . This can be also be achieved by $180 - 110.876 =$ awrt 69.1. $\theta = 1.2064...^c$ is A0.

Common response: $\cos\theta = \left(\frac{-12-24+12}{\sqrt{(-3)^2+(-12)^2+(6)^2}.\sqrt{(4)^2+(2)^2+(2)^2}}\right) = \frac{-24}{\sqrt{189}.\sqrt{24}}$ is M1A1...

Alternative Method: Vector Cross Product

Only apply this scheme if it is clear that a candidate is applying a vector cross product method.

$$\underline{\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}} = \left\{ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -2 \\ 2 & 1 & 1 \end{vmatrix} = 6\mathbf{i} - 5\mathbf{j} - 7\mathbf{k} \right\}$$

M1: Realisation that the vector cross product is required between $\pm A\mathbf{d}_1$ and $\pm B\mathbf{d}_2$. Allow one slip in $\mathbf{d}_1 = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

$$\sin\theta = \frac{\sqrt{(6)^2+(5)^2+(-7)^2}}{\sqrt{(1)^2+(4)^2+(-2)^2}.\sqrt{(2)^2+(1)^2+(1)^2}}$$

A1: Correct applied **equation**.

$$\sin\theta = \frac{\sqrt{110}}{\sqrt{21}.\sqrt{6}} \Rightarrow \theta = 69.1238974... = 69.1 \text{ (1 dp)}$$

A1: awrt 69.1

(c)

M1: Attempts to find $\overrightarrow{AP}$ in terms of the parameter by subtracting the components of $\overrightarrow{OP}$ from l_1 and $\overrightarrow{OA}$. Ignore the direction of subtraction and ignore any confusion between $\overrightarrow{OP}$ and $\overrightarrow{PO}$ or between $\overrightarrow{OA}$ and $\overrightarrow{AO}$. The correct subtraction of two components is enough to establish that subtraction is intended. The coordinates or position vector of P must be given in terms of a parameter. Taking $P:(x, y, z)$ gains no marks although this can be recovered later. See **Additional Solutions**.

A1: (M1 on epen) A correct expression for $\overrightarrow{AP}$. Again accept the reverse direction.

dM1: Depends on the previous M. Taking the scalar product of their expression for $\overrightarrow{AP}$ with $\mathbf{d}_1$ or a multiple of $\mathbf{d}_1$ and equating to 0 and obtaining an equation for λ. The equation must derive from an expression of the form $x_1x_2 + y_1y_2 + z_1z_2 = 0$. Differentiation can be used. See **Additional Solutions**.

A1: Solving to find $\lambda = \frac{1}{3}$.

ddM1: Depends on both previous Ms. Substitutes their value of the parameter into their expression for $\overrightarrow{OP}$. Substituting into $\overrightarrow{AP}$ is a common error which loses the mark.

Note: Needs 2 correct co-ordinates if $\lambda = \frac{1}{3}$ found and then P stated without method to gain ddM1.

	A1: $9\frac{1}{3}\mathbf{i}+14\frac{1}{3}\mathbf{j}-3\frac{2}{3}\mathbf{k}$. Accept vector notation or coordinates. ***Must be exact.***

7. (c)

Additional Solution 1:

Taking $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, in itself, can gain no marks but this may be converted to a parameter at a later stage in the solution and, at that stage, any relevant marks can be awarded.

For example, $\overrightarrow{AP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix} = \begin{pmatrix} x-4 \\ y-16 \\ z+3 \end{pmatrix}$

leading to: $\begin{pmatrix} x-4 \\ y-16 \\ z+3 \end{pmatrix} \square \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = x - 4 + 4y - 64 - 2z - 6 = 0$ No marks gained at this stage.

Using, $\overrightarrow{OP} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9+\lambda \\ 13+4\lambda \\ -3-2\lambda \end{pmatrix}$ on $x + 4y - 2z = 74$

which gives: $9 + \lambda + 4(13+4\lambda) - 2(-3-2\lambda) = 74$ At this stage award **M1A1** and **dM1** (which is implied by an equation)

$\Rightarrow \ 21\lambda + 67 = 74 \Rightarrow \underline{\lambda = \frac{1}{3}}$ A1: Solving to find $\lambda = \frac{1}{3}$.

Position vector

$\overrightarrow{OP} = \begin{pmatrix} 9 \\ 13 \\ -3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9\frac{1}{3} \\ 14\frac{1}{3} \\ -3\frac{2}{3} \end{pmatrix}$ or $\begin{pmatrix} \frac{28}{3} \\ \frac{43}{3} \\ -\frac{11}{3} \end{pmatrix}$ ddM1 A1

Additional Solution 2: Using Differentiation

$\overrightarrow{AP} = \begin{pmatrix} 9+\lambda \\ 13+4\lambda \\ -3-2\lambda \end{pmatrix} - \begin{pmatrix} 4 \\ 16 \\ -3 \end{pmatrix} = \begin{pmatrix} \lambda+5 \\ 4\lambda-3 \\ -2\lambda \end{pmatrix}$ **M1A1:** As main scheme

$AP^2 = (\lambda+5)^2 + (4\lambda-3)^2 + (-2\lambda)^2 = \left\{21\lambda^2 - 14\lambda + 34\right\}$

$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(AP^2\right) = 42\lambda - 14 = 0$ **M1**

leading to $\lambda = \frac{1}{3}$ **A1:** Solving to find $\lambda = \frac{1}{3}$.

... then apply the main scheme.

8. A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3 °C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ °C.

The rate of change of the temperature of the water in the bottle is modelled by the differential equation,

$$\frac{d\theta}{dt} = \frac{(3 - \theta)}{125}$$

(a) By solving the differential equation, show that,

$$\theta = Ae^{-0.008t} + 3$$

where A is a constant.

(4)

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16 °C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10 °C, giving your answer to the nearest minute.

(5)

Question Number	Scheme		Marks
8. (a)	$\left\{\frac{d\theta}{dt} = \frac{(3-\theta)}{125}\right\} \Rightarrow \int \frac{1}{3-\theta}\,d\theta = \int \frac{1}{125}\,dt$ or $\int \frac{125}{3-\theta}\,d\theta = \int dt$		B1
	$-\ln(\theta-3) = \frac{1}{125}t\ \{+\ c\}$ or $-\ln(3-\theta) = \frac{1}{125}t\ \{+\ c\}$	See notes.	M1 A1
	$\ln(\theta-3) = -\frac{1}{125}t\ + c$		
	$\theta - 3 = e^{-\frac{1}{125}t\ + c}$ or $e^{-\frac{1}{125}t}e^{c}$	***Correct*** completion to $\theta = Ae^{-0.008t}+3$.	
	$\theta = Ae^{-0.008t}+3$ *		A1
			[4]
(b)	$\{t=0\ ,\ \theta=16 \Rightarrow\}$ $16 = Ae^{-0.008(0)}+3$; $\Rightarrow \underline{A=13}$	***See notes.***	M1; A1
	$10 = 13e^{-0.008t}+3$	Substitutes $\theta=10$ into an equation **of the form** $\theta = Ae^{-0.008t}+3$, or equivalent. ***See notes.***	M1
	$e^{-0.008t} = \frac{7}{13} \Rightarrow -0.008t = \ln\left(\frac{7}{13}\right)$	Correct algebra to $-0.008t = \ln k$, where k is a positive value. ***See notes.***	M1
	$\left\{t = \frac{\ln\left(\frac{7}{13}\right)}{(-0.008)}\right\} = 77.3799... = 77$ (nearest minute)	awrt 77	A1
			[5] **9**

8. (a)

B1: (M1 on epen) Separates variables as shown. $d\theta$ and dt should be in the correct positions, though this mark can be implied by later working. Ignore the integral signs.

M1: Both $\pm\lambda\ln(3-\theta)$ or $\pm\lambda\ln(\theta-3)$ **and** $\pm\,\mu t$ where λ and μ are constants.

A1: For $-\ln(\theta-3) = \frac{1}{125}t$ or $-\ln(3-\theta) = \frac{1}{125}t$ or $-125\ln(\theta-3) = t$ or $-125\ln(3-\theta) = t$

Note: $+c$ is not needed for this mark.

A1: Correct completion to $\theta = Ae^{-0.008t}+3$. **Note:** $+c$ is needed for this mark.

Note: $\ln(\theta-3) = -\frac{1}{125}t\ + c$ leading to $\theta - 3 = e^{-\frac{1}{125}t} + e^{c}$ or $\theta - 3 = e^{-\frac{1}{125}t} + A$, would be final A0.

Note: From $-\ln(\theta-3) = \frac{1}{125}t\ + c$, then $\ln(\theta-3) = -\frac{1}{125}t\ + c$

$\Rightarrow \theta - 3 = e^{-\frac{1}{125}t\ + c}$ **or** $\theta - 3 = e^{-\frac{1}{125}t}e^{c} \Rightarrow \theta = Ae^{-0.008t}+3$ is required for A1.

Note: From $-\ln(3-\theta) = \frac{1}{125}t\ + c$, then $\ln(3-\theta) = -\frac{1}{125}t\ + c$

$\Rightarrow 3 - \theta = e^{-\frac{1}{125}t\ + c}$ **or** $3 - \theta = e^{-\frac{1}{125}t}e^{c} \Rightarrow \theta = Ae^{-0.008t}+3$ is sufficient for A1.

Note: The jump from $3 - \theta = Ae^{-\frac{1}{125}t}$ to $\theta = Ae^{-0.008t}+3$ is fine.

	Note: $\ln(\theta-3) = -\frac{1}{125}t + c \Rightarrow \theta - 3 = Ae^{-\frac{1}{125}t}$, where candidate writes $A = e^c$ is also acceptable.

8. (b)

M1: (B1 on epen) Substitutes $\theta = 16,\ t = 0$, into either their equation containing an unknown constant or the printed equation. **Note:** You can imply this method mark.

A1: (M1 on epen) $A = 13$. **Note:** $\theta = 13\mathrm{e}^{-0.008t} + 3$ without any working implies the first two marks, M1A1.

M1: Substitutes $\theta = 10$ into an equation **of the form** $\theta = A\mathrm{e}^{-0.008t} + 3$, or equivalent. where A is a positive or negative numerical value and A can be equal to 1 or -1.

M1: Uses correct algebra to rearrange **their equation** into the form $-0.008t = \ln k$, where k is ***a positive numerical value***.

A1: awrt 77 or awrt 1 hour 17 minutes.

<u>Alternative Method 1 for part (b)</u>

$\int \frac{1}{3-\theta}\,\mathrm{d}\theta = \int \frac{1}{125}\,\mathrm{d}t \Rightarrow -\ln(\theta - 3) = \frac{1}{125}t + c$	
$\{t = 0\ ,\ \theta = 16 \Rightarrow\}$ $-\ln(16-3) = \frac{1}{125}(0) + c$ $\Rightarrow c = -\ln 13$	**M1:** Substitutes $t = 0, \theta = 16$, into $-\ln(\theta - 3) = \frac{1}{125}t + c$ **A1:** $c = -\ln 13$
$-\ln(\theta - 3) = \frac{1}{125}t - \ln 13$ or $\ln(\theta - 3) = -\frac{1}{125}t + \ln 13$	
$-\ln(10-3) = \frac{1}{125}t - \ln 13$	**M1:** Substitutes $\theta = 10$ into an equation **of the form** $\pm\lambda\ln(\theta - 3) = \pm\frac{1}{125}t \pm \mu$ where λ, μ are numerical values.
$\ln 13 - \ln 7 = \frac{1}{125}t$	**M1:** Uses correct algebra to rearrange **their equation** into the form $\pm 0.008t = \ln C - \ln D$, where C, D are ***positive numerical values***.
$t = 77.3799... = 77$ (nearest minute)	**A1:** awrt 77.

<u>Alternative Method 2 for part (b)</u>

$\int \frac{1}{3-\theta}\,\mathrm{d}\theta = \int \frac{1}{125}\,\mathrm{d}t \Rightarrow -\ln\lvert 3 - \theta\rvert = \frac{1}{125}t + c$	
$\{t = 0\ ,\ \theta = 16 \Rightarrow\}$ $-\ln\lvert 3-16\rvert = \frac{1}{125}(0) + c$ $\Rightarrow c = -\ln 13$	**M1:** Substitutes $t = 0, \theta = 16$, into $-\ln(3 - \theta) = \frac{1}{125}t + c$ **A1:** $c = -\ln 13$
$-\ln\lvert 3 - \theta\rvert = \frac{1}{125}t - \ln 13$ or $\ln\lvert 3 - \theta\rvert = -\frac{1}{125}t + \ln 13$	
$-\ln(3-10) = \frac{1}{125}t - \ln 13$	**M1:** Substitutes $\theta = 10$ into an equation **of the form** $\pm\lambda\ln(3 - \theta) = \pm\frac{1}{125}t \pm \mu$ where λ, μ are numerical values.
$\ln 13 - \ln 7 = \frac{1}{125}t$	**M1:** Uses correct algebra to rearrange **their equation** into the form $\pm 0.008t = \ln C - \ln D$,

	$t \;=\; 77.3799... \;=\; 77 \;(\text{nearest minute})$	where *C*, *D* are ***positive numerical values***. **A1:** awrt 77.

8. (b)	***<u>Alternative Method 3 for part (b)</u>***	
	$\int_{16}^{10} \frac{1}{3-\theta}\,\mathrm{d}\theta \;=\; \int_0^t \frac{1}{125}\,\mathrm{d}t$	
	$= \left[-\ln\lvert 3-\theta\rvert\right]_{16}^{10} \;=\; \left[\frac{1}{125}t\right]_0^t$	**M1A1:** $\ln 13$ **M1:** Substitutes limit of $\theta = 10$ correctly.
	$-\ln 7 \;--\ln 13 \;= \frac{1}{125}t$	**M1:** Uses correct algebra to rearrange **their own equation** into the form $\pm 0.008t = \ln C - \ln D$, where *C*, *D* are ***positive numerical values***.
	$t \;=\; 77.3799... \;=\; 77 \;(\text{nearest minute})$	**A1:** awrt 77.
	<u>Alternative Method 4 for part (b)</u>	
	$\{\theta = 16 \Rightarrow\} \quad 16 = A\mathrm{e}^{-0.008t} + 3$ $\{\theta = 10 \Rightarrow\} \quad 10 = A\mathrm{e}^{-0.008t} + 3$	**M1*:** Writes down a pair of equations in *A* and *t* , for $\theta = 16$ and $\theta = 10$ with either *A* unknown or *A* being a positive or negative value. **A1:** Two equations with an unknown *A*.
	$-0.008t = \ln\left(\frac{13}{A}\right)$ or $-0.008t = \ln\left(\frac{7}{A}\right)$	**M1:** Uses ***correct algebra*** to solve both of **their equations** leading to answers of the form $-0.008t = \ln k$, where *k* is ***a positive numerical value***.
	$t_{(1)} = \frac{\ln\left(\frac{13}{A}\right)}{-0.008}$ and $t_{(2)} = \frac{\ln\left(\frac{7}{A}\right)}{-0.008}$	
	$t = t_{(1)} - t_{(2)} = \frac{\ln\left(\frac{13}{A}\right)}{-0.008} - \frac{\ln\left(\frac{7}{A}\right)}{-0.008}$	**M1:** Finds difference between the two times. (either way round).
	$\left\{t = \frac{\ln\left(\frac{7}{13}\right)}{(-0.008)}\right\} = \;77.3799... \;=\; 77 \;(\text{nearest minute})$	**A1:** awrt 77. Correct solution only.