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**Pearson Edexcel GCE**

| Centre Number | Candidate Number |
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# Core Mathematics C4

## Advanced

Friday 23 June 2017 – Morning
**Time: 1 hour 30 minutes**

Paper Reference
**6666/01**

**You must have:**
Mathematical Formulae and Statistical Tables (Pink)

Total Marks

**Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B). Coloured pencils and highlighter pens must not be used.
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- When a calculator is used, the answer should be given to an appropriate degree of accuracy.

## Information

- The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over*

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**1.** The curve $C$ has parametric equations

$$x = 3t - 4, \quad y = 5 - \frac{6}{t}, \qquad t > 0$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$

**(2)**

The point $P$ lies on $C$ where $t = \frac{1}{2}$

(b) Find the equation of the tangent to $C$ at the point $P$. Give your answer in the form $y = px + q$, where $p$ and $q$ are integers to be determined.

**(3)**

(c) Show that the cartesian equation for $C$ can be written in the form

$$y = \frac{ax + b}{x + 4}, \quad x > -4$$

where $a$ and $b$ are integers to be determined.

**(3)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **1.** | $x = 3t - 4,\ \ y = 5 - \dfrac{6}{t},\ \ \ t > 0$ | | |
| (a) | $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 3,\ \ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 6t^{-2}$ | | |
| | $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{6t^{-2}}{3}\ \left\{= \dfrac{6}{3t^2} = 2t^{-2} = \dfrac{2}{t^2}\right\}$ | their $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ divided by their $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ to give $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$ **or** their $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ multiplied by their $\dfrac{\mathrm{d}t}{\mathrm{d}x}$ to give $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$ | M1 |
| | | $\dfrac{6t^{-2}}{3}$, simplified or un-simplified, in terms of $t$. **See note.** | A1 isw |
| | Award **Special Case 1st M1** if **both** $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ **and** $\dfrac{\mathrm{d}y}{\mathrm{d}t}$ are stated **correctly** and **explicitly**. | | **[2]** |
| | **Note:** You can recover the work for part (a) in part (b). | | |
| (a) **Way 2** | $y = 5 - \dfrac{18}{x+4} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{18}{(x+4)^2} = \dfrac{18}{(3t)^2}$ | Writes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in the form $\dfrac{\pm\lambda}{(x+4)^2}$, ***and*** writes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ as a function of $t$. | M1 |
| | | Correct un-simplified or simplified answer, in terms of $t$. **See note.** | A1 isw |
| | | | **[2]** |
| (b) | $\left\{t = \dfrac{1}{2} \Rightarrow\right\} P\left(-\dfrac{5}{2}, -7\right)$ | $x = -\dfrac{5}{2},\ y = -7$ **or** $P\left(-\dfrac{5}{2}, -7\right)$ seen or implied. | B1 |
| | $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{\left(\frac{1}{2}\right)^2}$ **and** either<br>• $y - \text{"}-7\text{"} = \text{"}8\text{"}\left(x - \text{"}-\frac{5}{2}\text{"}\right)$<br>• $\text{"}-7\text{"} = (\text{"}8\text{"})(\text{"}-\frac{5}{2}\text{"}) + c$<br>So, $y = (\text{their } m_{\mathrm{T}})x + \text{"}c\text{"}$ | **Some** attempt to substitute $t = 0.5$ into their $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ which contains $t$ in order to find $m_{\mathrm{T}}$ **and either** applies $y - (\text{their } y_P) = (\text{their } m_{\mathrm{T}})(x - \text{their } x_P)$ **or** finds $c$ from $(\text{their } y_P) = (\text{their } m_{\mathrm{T}})(\text{their } x_P) + c$ and uses their numerical $c$ in $y = (\text{their } m_{\mathrm{T}})x + c$ | M1 |
| | **T**: $y = 8x + 13$ | $y = 8x + 13$ or $y = 13 + 8x$ | A1 cso |
| | **Note:** their $x_P$, their $y_P$ and their $m_T$ must be numerical values in order to award M1 | | **[3]** |
| (c) **Way 1** | $\left\{t = \dfrac{x+4}{3} \Rightarrow\right\} y = 5 - \dfrac{6}{\left(\dfrac{x+4}{3}\right)}$ | An attempt to eliminate $t$. **See notes.** | M1 |
| | | Achieves a correct equation in $x$ and $y$ only | A1 o.e. |
| | $\Rightarrow\ y = 5 - \dfrac{18}{x+4}\ \ \Rightarrow\ \ y = \dfrac{5(x+4) - 18}{x+4}$ | | |
| | So, $y = \dfrac{5x+2}{x+4},\ \{x > -4\}$ | $y = \dfrac{5x+2}{x+4}$ (or implied equation) | A1 cso |
| | | | **[3]** |
| (c) **Way 2** | $\left\{t = \dfrac{6}{5-y} \Rightarrow\right\} x = \dfrac{18}{5-y} - 4$ | An attempt to eliminate $t$. **See notes.** | M1 |
| | | Achieves a correct equation in $x$ and $y$ only | A1 o.e. |
| | $\Rightarrow\ (x+4)(5-y) = 18 \Rightarrow\ 5x - xy + 20 - 4y = 18$ | | |
| | $\{\Rightarrow\ 5x + 2 = y(x+4)\}$ So, $y = \dfrac{5x+2}{x+4},\ \{x > -4\}$ | $y = \dfrac{5x+2}{x+4}$ (or implied equation) | A1 cso |
| | | | **[3]** |
| | **Note:** Some or all of the work for part (c) can be recovered in part (a) or part (b) | | **8** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **1.** (c) **Way 3** | $y = \dfrac{3at - 4a + b}{3t - 4 + 4} = \dfrac{3at}{3t} - \dfrac{4a - b}{3t} = a - \dfrac{4a - b}{3t} \Rightarrow a = 5$ | A full method leading to the value of $a$ being found | M1 |
| | | $y = a - \dfrac{4a-b}{3t}$ and $a = 5$ | A1 |
| | $\dfrac{4a-b}{3} = 6 \Rightarrow b = 4(5) - 6(3) = 2$ | **Both** $a = 5$ **and** $b = 2$ | A1 |
| | | | **[3]** |

| | | **Question 1 Notes** |
|---|---|---|
| **1.** (a) | **Note** | Condone $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\left(\frac{6}{t^2}\right)}{3}$ for A1 |
| | **Note** | You can ignore subsequent working following on from a correct expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $t$. |
| (b) | **Note** | Using a changed gradient (i.e. applying $\dfrac{-1}{\text{their } \frac{\mathrm{d}y}{\mathrm{d}x}}$ or $\dfrac{1}{\text{their } \frac{\mathrm{d}y}{\mathrm{d}x}}$ or $-\left(\text{their } \frac{\mathrm{d}y}{\mathrm{d}x}\right)$) is M0. |
| | **Note** | **Final A1:** A correct solution is required from a correct $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. |
| | **Note** | **Final A1:** You can ignore subsequent working following on from a correct solution. |
| (c) | **Note** | **1st M1:** A full attempt to eliminate $t$ is defined as either<br>• rearranging one of the parametric equations to make $t$ the subject and substituting for $t$ in the other parametric equation (only the RHS of the equation required for M mark)<br>• rearranging both parametric equations to make $t$ the subject and putting the results equal to each other. |
| | **Note** | Award M1A1 for $\dfrac{6}{5-y} = \dfrac{x+4}{3}$ or equivalent. |

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**2.** $\mathrm{f}(x) = (2 + kx)^{-3}, \quad |kx| < 2$, where $k$ is a positive constant

The binomial expansion of $\mathrm{f}(x)$, in ascending powers of $x$, up to and including the term in $x^2$ is

$$A + Bx + \frac{243}{16}x^2$$

where $A$ and $B$ are constants.

(a) Write down the value of $A$. **(1)**

(b) Find the value of $k$. **(3)**

(c) Find the value of $B$. **(2)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **2.** | $\left\{(2+kx)^{-3} = 2^{-3}\left(1+\frac{kx}{2}\right)^{-3} = \frac{1}{8}\left(1+(-3)\left(\frac{kx}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(\frac{kx}{2}\right)^2+\ldots\right)\right\}, k>0$ | | |
| (a) | $\{A=\}\ \frac{1}{8}$ | $\frac{1}{8}$ or $2^{-3}$ or 0.125**,** clearly identified as $A$ **or** as their answer to part (a) | B1 **cao** |
| | | | **[1]** |
| (b) | $\left(\frac{1}{8}\right)\frac{(-3)(-4)}{2!}\left(\frac{k}{2}\right)^2$ | Uses the $x^2$ term of the binomial expansion to give **either** $\frac{(-3)(-4)}{2!}$ **or** $\left(\frac{k}{2}\right)^2$ **or** $\left(\frac{kx}{2}\right)^2$ **or** $\frac{(-3)(-4)}{2}$ **or** 6 | M1 |
| | | **either** (their $A$)$\frac{(-3)(-4)}{2!}\left(\frac{k}{2}\right)^2$ **or** (their $A$)$\frac{(-3)(-4)}{2!}\left(\frac{kx}{2}\right)^2$**,** where (their $A$) $\ne$ 1, **or** $\frac{3}{16}k^2$ **or** $\frac{3}{16}k^2x^2$ **or** $(2^{-5})\frac{(-3)(-4)}{2!}(kx)^2$ **or** $(2^{-5})\frac{(-3)(-4)}{2!}(k)^2$ | M1 o.e. |
| | $\left\{\text{So, } \left(\frac{1}{8}\right)\frac{(-3)(-4)}{2!}\left(\frac{k}{2}\right)^2 = \frac{243}{16} \Rightarrow \frac{3}{16}k^2 = \frac{243}{16} \Rightarrow k^2 = 81\right\}$ | | |
| | So, $k=9$ | $k=9$ cao | A1 **cso** |
| | **Note:** $k = \pm 9$ with no reference to $k=9$ only is A0 | | **[3]** |
| (c) | $\text{"}\left(\frac{1}{8}\right)\text{"}(-3)\left(\frac{k}{2}\right)$ | Uses the $x$ term of the binomial expansion to give **either** (their $A$)$(-3)\left(\frac{k}{2}\right)$ **or** (their $A$)$(-3)\left(\frac{kx}{2}\right)$; where (their $A$) $\ne$ 1, **or** $(2)^{-4}(-3)(k)$ **or** $(2)^{-4}(-3)(kx)$ **or** $-\frac{3k}{16}$ | M1 |
| | $\left\{\text{So, } B = \left(\frac{1}{8}\right)(-3)\left(\frac{9}{2}\right) \Rightarrow\right\}\ \underline{B = -\frac{27}{16}}$ | $-\frac{27}{16}$ or $-1\frac{11}{16}$ or $-1.6875$ | A1 **cso** |
| | | | **[2]** |
| | | | **6** |

**Question 2 Notes**

| | |
|---|---|
| **NOTE** | **IN THIS QUESTION IGNORE LABELLING AND MARK ALL PARTS TOGETHER.** |
| **Note** | $(2+kx)^{-3} = \frac{1}{8}\left(1-\frac{3}{2}kx+\frac{3}{2}k^2x^2+\ldots\right) = \frac{1}{8}-\frac{3}{16}kx+\frac{3}{16}k^2x^2+\ldots$ |
| **Note** | Writing down $\left\{\left(1+\frac{kx}{2}\right)^{-3}\right\} = 1+(-3)\left(\frac{kx}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(\frac{kx}{2}\right)^2+\ldots$ . gets (b) 1st M1 |
| **Note** | Writing down $\left\{(2+kx)^{-3}\right\} = \frac{1}{8}\left(1+(-3)\left(\frac{kx}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(\frac{kx}{2}\right)^2+\ldots\right)$ gets (b) 1st M1 2nd M1 and (c) M1 |
| **Note** | Writing down $\left\{(2+kx)^{-3}\right\} = 2^{-3}+(-3)(2^{-4})(kx)+\frac{(-3)(-4)}{2}(2^{-5})(kx)^2$ gets (b) 1st M1 2nd M1 and (c) M1 |
| **Note** | Writing down $\left\{(2+kx)^{-3}\right\} = (\text{their } A)\left(1+(-3)\left(\frac{kx}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(\frac{kx}{2}\right)^2+\ldots\right)$ where (their $A$) $\ne$ 1, gets (b) 1st M1 2nd M1 and (c) M1 |

| | | Question 2 Notes |
|---|---|---|
| **2.** (b), (c) | **Note** | $(\text{their } A)$ is defined as either<br>• their answer to part (a)<br>• their stated $A = ...$<br>• their "$2^{-3}$" in their stated $2^{-3}\left(1+\dfrac{kx}{2}\right)^{-3}$ |
| | **Note** | Give 2nd M0 in part (b) if $(\text{their } A) = 1$ |
| | **Note** | Give M0 in part (c) if $(\text{their } A) = 1$ |
| **2.** (c) | **Note** | Allow M1 for $(\text{their } A)(-3)\left(\dfrac{\text{their } k \text{ from (b)}}{2}\right)$ |
| | **Note** | Award A0 for $B = -\dfrac{27}{16}x$ |
| | **Note** | Allow A1 for $B = -\dfrac{27}{16}x$ followed by $B = -\dfrac{27}{16}$ **or** $-1\dfrac{11}{16}$ **or** $-1.6875$ |
| | **Note** | $k = -9$ leading to $B = \dfrac{27}{16}$ **or** $1\dfrac{11}{16}$ **or** $1.6875$ is A0 |
| | **Note** | Give A0 for finding both $B = -\dfrac{27}{16}$ and $B = \dfrac{27}{16}$ (without rejecting $B = \dfrac{27}{16}$) as their final answer. |
| | **Note** | The A1 mark in part (c) is for a correct solution only. |
| | **Note** | **Be careful!** It is possible to award M0A0 in part (c) for a solution leading to $B = -\dfrac{27}{16}$. E.g.<br>$\mathrm{f}(x) = (2+kx)^{-3} = 2^{-3}(1+kx)^{-3} = \dfrac{1}{8}\left(1+(-3)(kx)+\dfrac{(-3)(-4)}{2!}(kx)^2+...\right) = \dfrac{1}{8}-\dfrac{3k}{8}x+\dfrac{3k^2}{4}x^2+...$<br>leading to (a) $A = \dfrac{1}{8}$, (b) $k = \dfrac{9}{2}$, (c) $B = -\dfrac{27}{16}$, gets (a) B1, (b) M1M0A0 (c) M0A0 |
| **2.** (b), (c) | **Note** | ${}^{-3}C_0(2)^{-3} + {}^{-3}C_1(2)^{-4}(kx) + {}^{-3}C_2(2)^{-5}(kx)^2$ with the C terms not evaluated<br>gets (b) 1st M0 2nd M0 and (c) M0 |

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**3.**

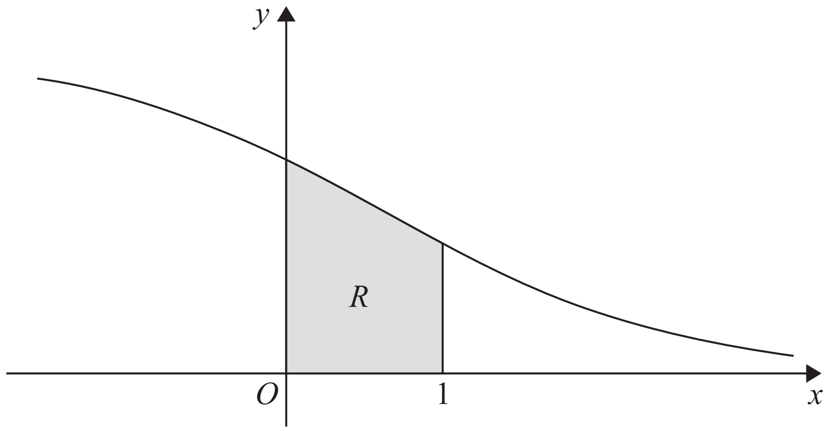


**Figure 1**

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{6}{(e^x + 2)}$, $x \in \mathbb{R}$

The finite region $R$, shown shaded in Figure 1, is bounded by the curve, the $y$-axis, the $x$-axis and the line with equation $x = 1$

The table below shows corresponding values of $x$ and $y$ for $y = \dfrac{6}{(e^x + 2)}$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y$ | 2 | | 1.71830 | 1.56981 | 1.41994 | 1.27165 |

(a) Complete the table above by giving the missing value of $y$ to 5 decimal places.
**(1)**

(b) Use the trapezium rule, with all the values of $y$ in the completed table, to find an estimate for the area of $R$, giving your answer to 4 decimal places.
**(3)**

(c) Use the substitution $u = e^x$ to show that the area of $R$ can be given by

$$\int_a^b \frac{6}{u(u+2)}\,du$$

where $a$ and $b$ are constants to be determined.
**(2)**

(d) Hence use calculus to find the exact area of $R$.
[*Solutions based entirely on graphical or numerical methods are not acceptable.*]
**(6)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| 3. | $x$: 0, 0.2, 0.4, 0.6, 0.8, 1; $y$: 2, ***1.8625426...***, 1.71830, 1.56981, 1.41994, 1.27165 | $y=\dfrac{6}{(2+e^x)}$ | |
| (a) | $\{\text{At } x=0.2,\}\ y=1.86254$ (5 dp) | 1.86254 | B1 **cao** |
| | **Note:** Look for this value on the given table or in their working. | | **[1]** |
| (b) | $\frac{1}{2}(0.2)\left[2+1.27165+2\left(\text{their } 1.86254+1.71830+1.56981+1.41994\right)\right]$ | Outside brackets $\frac{1}{2}\times(0.2)$ or $\frac{1}{10}$ or $\frac{1}{2}\times\frac{1}{5}$ | B1 o.e. |
| | | For structure of $[\ldots\ldots.]$ | M1 |
| | $\left\{=\frac{1}{10}(16.41283)\right\}=1.641283=1.6413$ (4 dp) | anything that rounds to 1.6413 | A1 |
| | | | **[3]** |
| (c) | $\{u=e^x \text{ or } x=\ln u \Rightarrow\}$ | | |
| | $\frac{du}{dx}=e^x$ **or** $\frac{du}{dx}=u$ **or** $\frac{dx}{du}=\frac{1}{u}$ **or** $du=u\,dx$ etc., **and** $\int\frac{6}{(e^x+2)}dx=\int\frac{6}{(u+2)u}du$ | **See notes** | B1 * |
| | $\{x=0\}\Rightarrow a=e^0\Rightarrow \underline{a=1}$<br>$\{x=1\}\Rightarrow b=e^1\Rightarrow \underline{b=e}$ | $a=1$ and $b=e$ or $b=e^1$ or evidence of $0\to1$ and $1\to e$ | B1 |
| | **NOTE: 1st B1 mark CANNOT be recovered for work in part (d)**<br>**NOTE: 2nd B1 mark CAN be recovered for work in part (d)** | | **[2]** |
| (d) **Way 1** | $\frac{6}{u(u+2)}\equiv\frac{A}{u}+\frac{B}{(u+2)}$<br>$\Rightarrow 6\equiv A(u+2)+Bu$ | Writing $\frac{6}{u(u+2)}\equiv\frac{A}{u}+\frac{B}{(u+2)}$, o.e. **or** $\frac{1}{u(u+2)}\equiv\frac{P}{u}+\frac{Q}{(u+2)}$, o.e., and a complete method for finding the value of at least one of **their** $A$ or **their** $B$ (or **their** $P$ or **their** $Q$) | M1 |
| | $u=0\Rightarrow A=3$<br>$u=-2\Rightarrow B=-3$ | Both **their** $A=3$ **and their** $B=-3$. (Or **their** $P=\frac{1}{2}$ **and their** $Q=-\frac{1}{2}$ with the factor of 6 in front of the integral sign) | A1 |
| | $\int\frac{6}{u(u+2)}du=\int\left(\frac{3}{u}-\frac{3}{(u+2)}\right)du$ | Integrates $\frac{M}{u}\pm\frac{N}{u\pm k}$, $M,N,k\neq 0$; (i.e. ***a two term partial fraction***) to obtain either $\pm\lambda\ln(\alpha u)$ **or** $\pm\mu\ln(\beta(u\pm k))$; $\lambda,\mu,\alpha,\beta\neq 0$ | M1 |
| | $=3\ln u-3\ln(u+2)$<br>or $=3\ln 2u-3\ln(2u+4)$ | Integration of both terms is **correctly followed through** from **their** $M$ and from **their** $N$. | A1 ft |
| | $\left\{\text{So}\left[3\ln u-3\ln(u+2)\right]_1^e\right\}$<br>$=\left(3\ln(e)-3\ln(e+2)\right)-\left(3\ln1-3\ln3\right)$<br>**[Note:** A proper consideration of the limit of $u=1$ is required for this mark] | **dependent on the 2nd M mark** Applies limits of e and 1 (or their $b$ and their $a$, where $b>0, b\neq 1, a>0$) in $u$ **or** applies limits of 1 and 0 in $x$ and subtracts the correct way round. | dM1 |
| | $=3-3\ln(e+2)+3\ln3$ **or** $3(1-\ln(e+2)+\ln3)$ **or** $3+3\ln\left(\frac{3}{e+2}\right)$<br>**or** $3\ln\left(\frac{e}{e+2}\right)-3\ln\left(\frac{1}{3}\right)$ **or** $3-3\ln\left(\frac{e+2}{3}\right)$ **or** $3\ln\left(\frac{3e}{e+2}\right)$ **or** $\ln\left(\frac{27e^3}{(e+2)^3}\right)$ | **see notes** | A1 **cso** |
| | **Note:** Allow $e^1$ in place of $e$ for the final A1 mark. | | **[6]** |
| | **Note:** Give final A0 for $3-3\ln e+2+3\ln3$ (i.e. bracketing error) unless recovered. | | **12** |
| | **Note:** Give final A0 for $3-3\ln(e+2)+3\ln3-3\ln1$, where $3\ln1$ has not been simplified to 0 | | |
| | **Note:** Give final A0 for $3\ln e-3\ln(e+2)+3\ln3$, where $3\ln e$ has not been simplified to 3 | | |

| | Question 3 Notes | |
|---|---|---|
| **3.** (b) | **Note** | **M1:** Do not allow an extra $y$-value ***or*** a repeated $y$ value in their […]<br>Do not allow an omission of a $y$-ordinate in their […] for M1 **unless** they give the correct answer of awrt 1.6413, in which case both M1 and A1 can be scored. |
| | **Note** | **A1:** Working must be seen to demonstrate the use of the trapezium rule.<br>(Actual area is 1.64150274…) |
| | **Note** | Full marks can be gained in part (b) for awrt 1.6413 even if B0 is given in part (a) |
| | **Note** | Award B1M1A1 for<br>$\frac{1}{10}(2+1.27165)+\frac{1}{5}(\text{their } 1.86254+1.71830+1.56981+1.41994) = \text{awrt } 1.6413$ |
| | **<u>Bracketing mistakes:</u> Unless the final answer implies that the calculation has been done correctly**<br>Award B1M0A0 for $\frac{1}{2}(0.2)+2+2(\text{their } 1.86254+1.71830+1.56981+1.41994)+1.27165\ \ (=16.51283)$<br>Award B1M0A0 for $\frac{1}{2}(0.2)(2+1.27165)+2(\text{their } 1.86254+1.71830+1.56981+1.41994)\ \ (=13.468345)$<br>Award B1M0A0 for $\frac{1}{2}(0.2)(2)+2(\text{their } 1.86254+1.71830+1.56981+1.41994)+1.27165\ \ (=14.61283)$ | |
| | **<u>Alternative method: Adding individual trapezia</u>**<br>$\text{Area} \approx 0.2\times\left[\frac{2+\text{"}1.86254\text{"}}{2}+\frac{\text{"}1.86254\text{"}+1.71830}{2}+\frac{1.71830+1.56981}{2}+\frac{1.56981+1.41994}{2}+\frac{1.41994+1.27165}{2}\right]$<br>$= 1.641283$<br>**B1** 0.2 and a divisor of 2 on all terms inside brackets<br>**M1** First and last ordinates once and two of the middle ordinates inside brackets ignoring the 2<br>**A1** anything that rounds to 1.6413 | |
| **3.** (c) | **1st B1** | Must start from either<br>• $\int y\,\mathrm{d}x$, with integral sign and $\mathrm{d}x$<br>• $\int \frac{6}{(e^x+2)}\,\mathrm{d}x$, with integral sign and $\mathrm{d}x$<br>• $\int \frac{6}{(e^x+2)}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$, with integral sign and $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$<br>**and** state either $\frac{\mathrm{d}u}{\mathrm{d}x}=e^x$ **or** $\frac{\mathrm{d}u}{\mathrm{d}x}=u$ **or** $\frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{u}$ **or** $\mathrm{d}u=u\,\mathrm{d}x$<br>**and** end at $\int \frac{6}{u(u+2)}\,\mathrm{d}u$, with integral sign and $\mathrm{d}u$, **with no incorrect working.** |
| | **Note** | So, just writing $\frac{\mathrm{d}u}{\mathrm{d}x}=e^x$ **and** $\int \frac{6}{(e^x+2)}\,\mathrm{d}x=\int \frac{6}{u(u+2)}\,\mathrm{d}u$ is sufficient for 1st B1 |
| | **Note** | Give 2nd B0 for $b = 2.718...$, without reference to $a=1$ and $b=e$ or $b=e^1$ |
| | **Note** | You can also give the 1st B1 mark for using a reverse process. i.e.<br>Proceeding from $\int \frac{6}{u(u+2)}\,\mathrm{d}u$ to $\int \frac{6}{(e^x+2)}\,\mathrm{d}x$, **with no incorrect working**,<br>and stating either $\frac{\mathrm{d}u}{\mathrm{d}x}=e^x$ **or** $\frac{\mathrm{d}u}{\mathrm{d}x}=u$ **or** $\frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{u}$ **or** $\mathrm{d}u=u\,\mathrm{d}x$ |
| **3.** (d) | **Note** | Give final A0 for $3-3\ln(e+2)+3\ln 3$ simplifying to $1-\ln(e+2)+\ln 3$<br>(i.e. dividing their correct final answer by 3)<br>Otherwise, you can ignore incorrect working (isw) following on from a correct exact value. |
| | **Note** | A decimal answer of 1.641502724... (without a correct **exact** answer) is final A0 |
| | **Note** | $\left[-3\ln(u+2)+3\ln u\right]_1^e$ followed by awrt 1.64 (without a correct **exact** answer) is final M1A0 |

| | | **Question 3 Notes Continued** |
|---|---|---|
| **3.** (d) | **Note** | **BE CAREFUL! Candidates will assign their own "*A*" and "*B*" for this question.** |
| | **Note** | ***Writing down*** $\frac{6}{(u+2)u}$ in the form $\frac{A}{(u+2)} + \frac{B}{u}$ with at least one of *A* or *B* correct is 1st M1 |
| | **Note** | ***Writing down*** $\frac{6}{(u+2)u}$ as $\frac{-3}{(u+2)} + \frac{3}{u}$ is 1st M1 1st A1. |
| | **Note** | **Condone** $\int \left(\frac{3}{u} - \frac{3}{(u+2)}\right) \mathrm{d}u$ to give $3\ln u - 3\ln u + 2$ (poor bracketing) for 2nd A1. |
| | **Note** | **Award M0A0M1A1ft** for a candidate who writes down<br>e.g. $\int \frac{6}{u(u+2)}\,\mathrm{d}u = \int \left(\frac{6}{u} + \frac{6}{(u+2)}\right) \mathrm{d}u = 6\ln u + 6\ln(u+2)$<br>**AS EVIDENCE OF WRITING** $\frac{6}{u(u+2)}$ **AS PARTIAL FRACTIONS.** |
| | **Note** | **Award M0A0M0A0** for a candidate who writes down<br>$\int \frac{6}{u(u+2)}\,\mathrm{d}u = 6\ln u + 6\ln(u+2)$ **or** $\int \frac{6}{u(u+2)}\,\mathrm{d}u = \ln u + 6\ln(u+2)$<br>**WITHOUT ANY EVIDENCE OF WRITING** $\frac{6}{u(u+2)}$ as partial fractions. |
| | **Note** | **Award M1A1M1A1** for a candidate who writes down<br>$\int \frac{6}{u(u+2)}\,\mathrm{d}u = 3\ln u - 3\ln(u+2)$<br>**WITHOUT ANY EVIDENCE OF WRITING** $\frac{6}{u(u+2)}$ as partial fractions. |
| | **Note** | If they lose the "6" and find $\int_1^e \frac{1}{u(u+2)}\mathrm{d}u$ we can allow a maximum of M1A0M1A1ftM1A0 |

| | | | |
|---|---|---|---|
| | **Question 3 Notes Continued** | | |
| **3.** (d) **Way 2** | $\left\{\int \frac{6}{u^2+2u}\,\mathrm{d}u = \int \frac{3(2u+2)}{u^2+2u}\,\mathrm{d}u - \int \frac{6u}{u^2+2u}\,\mathrm{d}u\right\}$ | | |
| | $= \int \frac{3(2u+2)}{u^2+2u}\,\mathrm{d}u - \int \frac{6}{u+2}\,\mathrm{d}u$ | $\int \frac{\pm\alpha(2u+2)}{u^2+2u}\{\mathrm{d}u\} \pm \int \frac{\delta}{u+2}\{\mathrm{d}u\}$, $\alpha, \beta, \delta \neq 0$ | M1 |
| | | Correct expression | A1 |
| ` | $= 3\ln(u^2+2u) - 6\ln(u+2)$ | Integrates $\frac{\pm M(2u+2)}{u^2+2u} \pm \frac{N}{u \pm k}$, $M, N, k \neq 0$, to obtain any one of $\pm\lambda\ln(u^2+2u)$ **or** $\pm\mu\ln(\beta(u\pm k))$; $\lambda, \mu, \beta \neq 0$ | M1 |
| | | Integration of both terms is **correctly followed through** from **their** $M$ and from **their** $N$ | A1 ft |
| | $\left\{\text{So, } \left[3\ln(u^2+2u) - 6\ln(u+2)\right]_1^{\mathrm{e}}\right\}$<br>$= \left(3\ln(\mathrm{e}^2+2\mathrm{e}) - 6\ln(\mathrm{e}+2)\right) - \left(3\ln 3 - 6\ln 3\right)$ | **dependent on the 2nd M mark** Applies limits of e and 1 (or their $b$ and their $a$, where $b > 0, b \neq 1, a > 0$) in $u$ **or** applies limits of 1 and 0 in $x$ and subtracts the correct way round. | dM1 |
| | $= 3\ln(\mathrm{e}^2+2\mathrm{e}) - 6\ln(\mathrm{e}+2) + 3\ln 3$ | $3\ln(\mathrm{e}^2+2\mathrm{e}) - 6\ln(\mathrm{e}+2) + 3\ln 3$ | A1 o.e. |
| | | | **[6]** |
| **3.** (d) **Way 3** | Applying $u = \theta - 1$ | | |
| | $\left\{\int_1^{\mathrm{e}} \frac{6}{u(u+2)}\,\mathrm{d}u = \right\} \int_2^{1+\mathrm{e}} \frac{6}{(\theta-1)(\theta+1)}\,\mathrm{d}\theta = \int_2^{1+\mathrm{e}} \frac{6}{\theta^2-1}\,\mathrm{d}u = \left[3\ln\left(\frac{\theta-1}{\theta+1}\right)\right]_2^{1+\mathrm{e}}$ | | M1A1M1A1 |
| | $= 3\ln\left(\frac{1+\mathrm{e}-1}{\mathrm{e}+1+1}\right) - 3\ln\left(\frac{2-1}{2+1}\right) = 3\ln\left(\frac{\mathrm{e}}{\mathrm{e}+2}\right) - 3\ln\left(\frac{1}{3}\right)$ | 3rd M mark is dependent on 2nd M mark | dM1A1 |
| | | | **[6]** |

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**4.** The curve $C$ has equation

$$4x^2 - y^3 - 4xy + 2^y = 0$$

The point $P$ with coordinates $(-2, 4)$ lies on $C$.

(a) Find the exact value of $\frac{dy}{dx}$ at the point $P$.
**(6)**

The normal to $C$ at $P$ meets the $y$-axis at the point $A$.

(b) Find the $y$ coordinate of $A$, giving your answer in the form $p + q\ln 2$, where $p$ and $q$ are constants to be determined.
**(3)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **4.** | $4x^2 - y^3 - 4xy + 2^y = 0$ | | |
| (a) **Way 1** | $\left\{\cancel{\frac{dy}{dx}} \cancel{=}\right\} \underline{8x - 3y^2\frac{dy}{dx}} \; \underline{\underline{-4y - 4x\frac{dy}{dx}}} + \overline{\overline{2^y \ln 2\frac{dy}{dx}}} \underline{= 0}$ | | M1 $\underline{\text{A1}}$ $\underline{\underline{\text{M1}}}$ $\overline{\overline{\text{B1}}}$ |
| | $8(-2) - 3(4)^2\frac{dy}{dx} - 4(4) - 4(-2)\frac{dy}{dx} + 2^4\ln 2\frac{dy}{dx} = 0$ | **dependent on the first M mark** | dM1 |
| | $-16 - 48\frac{dy}{dx} - 16 + 8\frac{dy}{dx} + 16\ln 2\frac{dy}{dx} = 0$ | | |
| | $\frac{dy}{dx} = \frac{32}{-40+16\ln 2}$ or $\frac{-32}{40-16\ln 2}$ or $\frac{4}{-5+2\ln 2}$ or $\frac{4}{-5+\ln 4}$ | or exact equivalent | A1 **cso** |
| | **NOTE: You can recover work for part (a) in part (b)** | | **[6]** |
| (b) | e.g. $m_N = \frac{-40+16\ln 2}{-32}$ or $\frac{40-16\ln 2}{32}$ | Applying $m_N = \frac{-1}{m_T}$ to find a numerical $m_N$ <br> **Can be implied by later working** | M1 |
| | • $y - 4 = \left(\frac{40-16\ln 2}{32}\right)(x - -2)$ <br> Cuts $y$-axis $\Rightarrow x = 0 \Rightarrow y - 4 = \left(\frac{40-16\ln 2}{32}\right)(2)$ <br> • $4 = \left(\frac{40-16\ln 2}{32}\right)(-2) + c$ | Using a numerical $m_N\ (\neq m_T)$, either $y - 4 = m_N(x - -2)$ **and** sets $x = 0$ in their normal equation **or** $4 = (\text{their } m_N)(-2) + c$ | M1 |
| | $\left\{\Rightarrow c = 4 + \frac{40-16\ln 2}{16}, \text{ so } y = \frac{104-16\ln 2}{16} \Rightarrow\right\}$ | | |
| | $y$ (or $c$) $= \frac{13}{2} - \ln 2$ | $\frac{104}{16} - \ln 2$ **or** $\frac{13}{2} - \ln 2$ **or** $-\ln 2 + \frac{13}{2}$ | A1 **cso isw** |
| | **Note:** Allow exact equivalents in the form $p - \ln 2$ for the final A mark | | **[3]** |
| | | | **9** |
| (a) **Way 2** | $\left\{\cancel{\frac{dx}{dy}} \cancel{=}\right\} \underline{8x\frac{dx}{dy} - 3y^2} \; \underline{\underline{-4y\frac{dx}{dy} - 4x}} + \overline{\overline{2^y \ln 2}} \underline{= 0}$ | | M1 $\underline{\text{A1}}$ $\underline{\underline{\text{M1}}}$ $\overline{\overline{\text{B1}}}$ |
| | $8(-2)\frac{dx}{dy} - 3(4)^2 - 4(4)\frac{dx}{dy} - 4(-2) + 2^4\ln 2 = 0$ | **dependent on the first M mark** | dM1 |
| | $\frac{dy}{dx} = \frac{32}{-40+16\ln 2}$ or $\frac{-32}{40-16\ln 2}$ or $\frac{4}{-5+2\ln 2}$ or $\frac{4}{-5+\ln 4}$ | or exact equivalent | A1 **cso** |
| | **Note: You must be clear that Way 2 is being applied before you use this scheme** | | **[6]** |

| | | |
|---|---|---|
| | **Question 4 Notes** | |
| 4. (a) | **Note** | <u>**For the first four marks**</u> |
| | | Writing down ***from no working*** <br> • $\frac{dy}{dx} = \frac{4y - 8x}{-3y^2 - 4x + 2^y\ln 2}$ or $\frac{8x - 4y}{3y^2 + 4x - 2^y\ln 2}$ scores M1A1M1B1 <br> • $\frac{dy}{dx} = \frac{8x - 4y}{-3y^2 - 4x + 2^y\ln 2}$ or $\frac{4y - 8x}{3y^2 + 4x - 2^y\ln 2}$ scores M1A0M1B1 |
| | | Writing $8x\,dx - 3y^2\,dy - 4y\,dx - 4x\,dy + 2^y\ln 2\,dy = 0$ scores M1A1M1B1 |

| | **Question 4 Notes Continued** | |
|---|---|---|
| **4.** (a) | **1st M1** | Differentiates implicitly to include ***either*** $\pm 4x\frac{dy}{dx}$ ***or*** $-y^3 \to \pm\lambda y^2\frac{dy}{dx}$ ***or*** $2^y \to \pm m2^y\frac{dy}{dx}$ <br> (Ignore $\left(\frac{dy}{dx}=\right)$). $\lambda$, $m$ are constants which can be 1 |
| | **1st A1** | **Both** $4x^2 - y^3 \to 8x - 3y^2\frac{dy}{dx}$ **and** $=0 \to =0$ |
| | **Note** | e.g. $8x - 3y^2\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} + 2^y\ln 2\frac{dy}{dx} \to -3y^2\frac{dy}{dx} - 4x\frac{dy}{dx} + 2^y\ln 2\frac{dy}{dx} = 4y - 8x$ <br> **or** e.g. $-16 - 48\frac{dy}{dx} - 16 + 8\frac{dy}{dx} + 16\ln 2\frac{dy}{dx} \to -48\frac{dy}{dx} + 8\frac{dy}{dx} + 16\ln 2\frac{dy}{dx} = 32$ <br> will get 1st A1 (implied) as the "$=0$" can be implied by the rearrangement of their equation. |
| | **2nd M1** | $-4xy \to -4y - 4x\frac{dy}{dx}$ or $4y - 4x\frac{dy}{dx}$ or $-4y + 4x\frac{dy}{dx}$ or $4y + 4x\frac{dy}{dx}$ |
| | **B1** | $2^y \to 2^y\ln 2\frac{dy}{dx}$ **or** $2^y \to e^{y\ln 2}\ln 2\frac{dy}{dx}$ |
| | **Note** | If an extra term appears then award 1st A0 |
| | **3rd dM1** <br><br> **Note** | **dependent on the first M mark** <br> For substituting $x=-2$ and $y=4$ into an equation involving $\frac{dy}{dx}$ <br> M1 can be gained by seeing at least one example of substituting $x=-2$ and at least one example of substituting $y=4$ **unless** it is clear that they are instead applying $x=4$ and $y=-2$ <br> Otherwise, you will NEED to check (with your calculator) that $x=-2$, $y=4$ that has been substituted into their equation involving $\frac{dy}{dx}$ |
| | **Note** | **A1 cso:** If the candidate's solution is not completely correct, then do not give this mark. |
| | **Note** | **isw:** You can, however, ignore subsequent working following on from correct solution. |
| (b) | **Note** | The 2nd M1 mark can be implied by later working. <br> **Eg. Award 1st M1 and 2nd M1** for $\frac{y-4}{2} = \frac{-1}{\text{their } m_T \text{ evaluated at } x=-2 \text{ and } y=4}$ |
| | **Note** | **A1:** Allow the alternative answer $\{y=\}\ \ln\left(\frac{1}{2}\right) + \frac{13}{2\ln 2}(\ln 2)$ which is in the form $p + q\ln 2$ |
| **4.** (a) **Way 2** | **1st M1** | Differentiates implicitly to include ***either*** $\pm 4y\frac{dx}{dy}$ ***or*** $4x^2 \to \pm\lambda x\frac{dx}{dy}$ <br> (Ignore $\left(\frac{dx}{dy}=\right)$). $\lambda$ is a constant which can be 1 |
| | **1st A1** | **Both** $4x^2 - y^3 \to 8x\frac{dx}{dy} - 3y^2$ **and** $=0 \to =0$ |
| | **2nd M1** | $-4xy \to -4y\frac{dx}{dy} - 4x$ or $4y\frac{dx}{dy} - 4x$ or $-4y\frac{dx}{dy} + 4x$ or $4y\frac{dx}{dy} + 4x$ |
| | **B1** | $2^y \to 2^y\ln 2$ |
| | **3rd dM1** | **dependent on the first M mark** <br> For substituting $x=-2$ and $y=4$ into an equation involving $\frac{dx}{dy}$ |

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**5.**

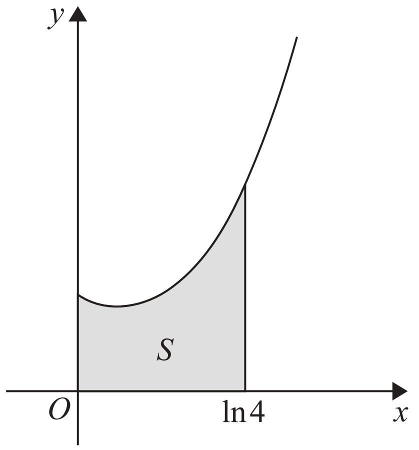


Diagram not drawn to scale

**Figure 2**

The finite region $S$, shown shaded in Figure 2, is bounded by the $y$-axis, the $x$-axis, the line with equation $x = \ln 4$ and the curve with equation

$$y = e^x + 2e^{-x}, \quad x \geqslant 0$$

The region $S$ is rotated through $2\pi$ radians about the $x$-axis.

Use integration to find the exact value of the volume of the solid generated. Give your answer in its simplest form.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

**(7)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **5.** **Way 1** | $y = e^{x} + 2e^{-x},\ x \geq 0$ | | |
| | $\{V =\}\ \pi\int_0^{\ln 4}\left(e^{x} + 2e^{-x}\right)^2 dx$ | For $\pi\int\left(e^{x} + 2e^{-x}\right)^2$. Ignore limits and $dx$. Can be implied. | B1 |
| | $= \{\pi\}\int_0^{\ln 4}\left(e^{2x} + 4e^{-2x} + 4\right)dx$ | Expands $\left(e^{x} + 2e^{-x}\right)^2 \to \pm\alpha e^{2x} \pm \beta e^{-2x} \pm \delta$ where $\alpha, \beta, \delta \neq 0$. Ignore $\pi$, integral sign, limits and $dx$. This can be implied by later work. | M1 |
| | $= \{\pi\}\left[\frac{1}{2}e^{2x} - 2e^{-2x} + 4x\right]_0^{\ln 4}$ | Integrates at least one of either $\pm\alpha e^{2x}$ to give $\pm\frac{\alpha}{2}e^{2x}$ or $\pm\beta e^{-2x}$ to give $\pm\frac{\beta}{2}e^{-2x}$ $\alpha, \beta \neq 0$ | M1 |
| | | **dependent on the 2nd M mark** $e^{2x} + 4e^{-2x} \to \frac{1}{2}e^{2x} - 2e^{-2x}$, which can be simplified or un-simplified | A1 |
| | | $4 \to 4x$ or $4e^{0}x$ | B1 cao |
| | $= \{\pi\}\left(\left(\frac{1}{2}e^{2(\ln 4)} - 2e^{-2(\ln 4)} + 4(\ln 4)\right) - \left(\frac{1}{2}e^{0} - 2e^{0} + 4(0)\right)\right)$ | **dependent on the previous method mark.** Some evidence of applying limits of $\ln 4$ o.e. and 0 to a changed function in $x$ and subtracts the correct way round. **Note:** A proper consideration of the limit of 0 is required. | dM1 |
| | $= \{\pi\}\left(\left(8 - \frac{1}{8} + 4\ln 4\right) - \left(\frac{1}{2} - 2\right)\right)$ | | |
| | $= \frac{75}{8}\pi + 4\pi\ln 4$ **or** $\frac{75}{8}\pi + 8\pi\ln 2$ **or** $\pi\left(\frac{75}{8} + 4\ln 4\right)$ **or** $\pi\left(\frac{75}{8} + 8\ln 2\right)$ **or** $\frac{75}{8}\pi + \ln 2^{8\pi}$ **or** $\frac{75}{8}\pi + \pi\ln 256$ **or** $\ln\left(2^{8\pi}e^{\frac{75}{8}\pi}\right)$ **or** $\frac{1}{8}\pi\left(75 + 32\ln 4\right)$, etc | | A1 isw |
| | | | **[7]** |
| | | | **7** |

**Question 5 Notes**

| | | |
|---|---|---|
| **5.** | **Note** | $\pi$ is only required for the 1st B1 mark and the final A1 mark. |
| | **Note** | Give 1st B0 for writing $\pi\int y^2\, dx$ followed by $2\pi\int\left(e^{x} + 2e^{-x}\right)^2 dx$ |
| | **Note** | Give 1st M1 for $\left(e^{x} + 2e^{-x}\right)^2 \to e^{2x} + 4e^{-2x} + 2e^{0} + 2e^{0}$ because $\delta = 2e^{0} + 2e^{0}$ |
| | **Note** | A decimal answer of 46.8731... or $\pi(14.9201...)$ (without a correct **exact** answer) is A0 |
| | **Note** | $\pi\left[\frac{1}{2}e^{2x} - 2e^{-2x} + 4x\right]_0^{\ln 4}$ followed by awrt 46.9 (without a correct **exact** answer) is final dM1A0 |
| | **Note** | Allow exact equivalents which should be in the form $a\pi + b\pi\ln c$ or $\pi(a + b\ln c)$, where $a = \frac{75}{8}$ or $9\frac{3}{8}$ or 9.375. Do not allow $a = \frac{150}{16}$ or $9\frac{6}{16}$ |
| | **Note** | Give B1M0M1A1B0M1A0 for the common response $\pi\int_0^{\ln 4}\left(e^{x} + 2e^{-x}\right)^2 dx \to \pi\int_0^{\ln 4}\left(e^{2x} + 4e^{-2x}\right)dx = \pi\left[\frac{1}{2}e^{2x} - 2e^{-2x}\right]_0^{\ln 4} = \frac{75}{8}\pi$ |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **5.** **Way 2** | $y = e^x + 2e^{-x},\ x \geq 0$ | | |
| | $\{V =\}\ \pi \int_0^{\ln 4} \left(e^x + 2e^{-x}\right)^2 \mathrm{d}x$ | For $\pi \int \left(e^x + 2e^{-x}\right)^2$ Ignore limits and $\mathrm{d}x$. Can be implied. | B1 |
| | $u = e^x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = e^x = u$ and $x = \ln 4 \Rightarrow u = 4,\ x = 0 \Rightarrow u = e^0 = 1$ | | |
| | $V = \{\pi\} \int_1^4 \left(u + \dfrac{2}{u}\right)^2 \dfrac{1}{u}\mathrm{d}u = \{\pi\} \int_1^4 \left(u^2 + \dfrac{4}{u^2} + 4\right)\dfrac{1}{u}\mathrm{d}u$ | | |
| | $= \{\pi\} \int_1^4 \left(u + \dfrac{4}{u^3} + \dfrac{4}{u}\right)\mathrm{d}u$ | $\left(e^x + 2e^{-x}\right)^2 \to \pm\alpha u \pm \beta u^{-3} \pm \delta u^{-1}$ where $u = e^x,\ \alpha, \beta, \delta \neq 0$. Ignore $\pi$, integral sign, limits and $\mathrm{d}u$. This can be implied by later work. | M1 |
| | $= \{\pi\} \left[\dfrac{1}{2}u^2 - \dfrac{2}{u^2} + 4\ln u\right]_1^4$ | Integrates at least one of either $\pm\alpha u$ to give $\pm\dfrac{\alpha}{2}u^2$ or $\pm\beta u^{-3}$ to give $\pm\dfrac{\beta}{2}u^{-2}$ $\alpha, \beta \neq 0$, where $u = e^x$ | M1 |
| | | **dependent on the 2nd M mark** $u + 4u^{-3} \to \dfrac{1}{2}u^2 - 2u^{-2}$, simplified or un-simplified, where $u = e^x$ | A1 |
| | | $4u^{-1} \to 4\ln u$, where $u = e^x$ | B1 cao |
| | $= \{\pi\}\left(\left(\dfrac{1}{2}(4)^2 - \dfrac{2}{(4)^2} + 4\ln 4\right) - \left(\dfrac{1}{2}(1)^2 - \dfrac{2}{(1)^2} + 4\ln 1\right)\right)$ | **dependent on the previous method mark.** Some evidence of applying limits of 4 and 1 to a changed function in $u$ [or $\ln 4$ o.e. and 0 to an integrated function in $x$] and subtracts the correct way round. | dM1 |
| | $= \{\pi\}\left(\left(8 - \dfrac{1}{8} + 4\ln 4\right) - \left(\dfrac{1}{2} - 2\right)\right)$ | | |
| | $= \dfrac{75}{8}\pi + 4\pi\ln 4$ **or** $\dfrac{75}{8}\pi + 8\pi\ln 2$ **or** $\pi\left(\dfrac{75}{8} + 4\ln 4\right)$ **or** $\pi\left(\dfrac{75}{8} + 8\ln 2\right)$ **or** $\dfrac{75}{8}\pi + \ln 2^{8\pi}$ **or** $\dfrac{75}{8}\pi + \pi\ln 256$ **or** $\ln\left(2^{8\pi}e^{\frac{75}{8}\pi}\right)$ **or** $\dfrac{1}{8}\pi\left(75 + 32\ln 4\right)$, etc | | A1 isw |
| | | | **[7]** |

Leave blank

**6.** With respect to a fixed origin $O$, the lines $l_1$ and $l_2$ are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$$

where $\lambda$ and $\mu$ are scalar parameters.

The lines $l_1$ and $l_2$ intersect at the point $X$.

(a) Find the coordinates of the point $X$.
**(3)**

(b) Find the size of the acute angle between $l_1$ and $l_2$, giving your answer in degrees to 2 decimal places.
**(3)**

The point $A$ lies on $l_1$ and has position vector $\begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix}$

(c) Find the distance $AX$, giving your answer as a surd in its simplest form.
**(2)**

The point $Y$ lies on $l_2$. Given that the vector $\overrightarrow{YA}$ is perpendicular to the line $l_1$

(d) find the distance $YA$, giving your answer to one decimal place.
**(2)**

The point $B$ lies on $l_1$ where $|\overrightarrow{AX}| = 2|\overrightarrow{AB}|$.

(e) Find the two possible position vectors of $B$.
**(3)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **6.** | $l_1 : \mathbf{r} = \begin{pmatrix}4\\28\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\-5\\1\end{pmatrix}$, $l_2 : \mathbf{r} = \begin{pmatrix}5\\3\\1\end{pmatrix} + \mu\begin{pmatrix}3\\0\\-4\end{pmatrix}$ ; $\overrightarrow{OA} = \begin{pmatrix}2\\18\\6\end{pmatrix}$ lies on $l_1$ | Let $\theta_{\text{Acute}}$ be the acute angle between $l_1$ and $l_2$ | |
| (a) | $\{l_1 = l_2 \Rightarrow\}\ 28 - 5\lambda = 3\ \{\Rightarrow \lambda = 5\}$ or $4 - \lambda = 5 + 3\mu$ and $4 + \lambda = 1 - 4\mu\ \{\Rightarrow \mu = -2\}$ | $28 - 5\lambda = 3$ or $4 - \lambda = 5 + 3\mu$ and $4 + \lambda = 1 - 4\mu$ or $\lambda = 5$ or $\mu = -2$ (Can be implied). | B1 |
| | $\{\overrightarrow{OX} =\} \begin{pmatrix}4\\28\\4\end{pmatrix} + 5\begin{pmatrix}-1\\-5\\1\end{pmatrix}$ or $\begin{pmatrix}5\\3\\1\end{pmatrix} - 2\begin{pmatrix}3\\0\\-4\end{pmatrix}$ | Puts $l_1 = l_2$ and solves to find $\lambda$ and/or $\mu$ **and** substitutes their value for $\lambda$ into $l_1$ or their value for $\mu$ into $l_2$ | M1 |
| | So, $X(-1, 3, 9)$ | $(-1, 3, 9)$ or $\begin{pmatrix}-1\\3\\9\end{pmatrix}$ or $-\mathbf{i} + 3\mathbf{j} + 9\mathbf{k}$ or condone $\begin{matrix}-1\\3\\9\end{matrix}$ | A1 **cao** |
| | | | **[3]** |
| (b) **Way 1** | $\mathbf{d}_1 = \begin{pmatrix}-1\\-5\\1\end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix}3\\0\\-4\end{pmatrix} \Rightarrow \begin{pmatrix}-1\\-5\\1\end{pmatrix} \bullet \begin{pmatrix}3\\0\\-4\end{pmatrix}$ | Realisation that the dot product is required between $\mathbf{d}_1$ and $\mathbf{d}_2$ or a multiple of $\mathbf{d}_1$ and $\mathbf{d}_2$ | M1 |
| | $\cos\theta = \dfrac{\pm\begin{pmatrix}-1\\-5\\1\end{pmatrix} \bullet \begin{pmatrix}3\\0\\-4\end{pmatrix}}{\sqrt{(-1)^2 + (-5)^2 + (1)^2}\,.\sqrt{(3)^2 + (0)^2 + (-4)^2}}$ $\left\{= \dfrac{-7}{\sqrt{27}\,.\sqrt{25}}\right\}$ | **dependent on the 1st M mark.** Applies dot product formula between $\mathbf{d}_1$ and $\mathbf{d}_2$ or a multiple of $\mathbf{d}_1$ and $\mathbf{d}_2$ | dM1 |
| | $\{\theta = 105.6303588... \Rightarrow\}\ \theta_{\text{Acute}} = 74.36964117... = 74.37$ (2 dp) | awrt 74.37 seen in (b) only | A1 |
| | | | **[3]** |
| (c) | $\overrightarrow{AX} = "\overrightarrow{OX}" - \overrightarrow{OA} = \begin{pmatrix}-1\\3\\9\end{pmatrix} - \begin{pmatrix}2\\18\\6\end{pmatrix} = \begin{pmatrix}-3\\-15\\3\end{pmatrix}$ or $A_{\lambda=2}, X_{\lambda=5} \Rightarrow AX = 3\lvert\mathbf{d}_1\rvert$, $\left\{\lvert\mathbf{d}_1\rvert = \sqrt{27}\right\}$ | | |
| | $AX = \sqrt{(-3)^2 + (-15)^2 + (3)^2}$ or $3\sqrt{27}$ $\left\{= \sqrt{243}\right\} = 9\sqrt{3}$ | Full method for finding $AX$ or $XA$ | M1 |
| | | $9\sqrt{3}$ seen in (c) only | A1 **cao** |
| | **Note:** You cannot recover work for part (c) in either part (d) or part (e). | | **[2]** |
| (d) **Way 1** | $\dfrac{YA}{"9\sqrt{3}"} = \tan("74.36964...")$ | $\dfrac{YA}{\text{their } \lvert\overrightarrow{AX}\rvert} = \tan\theta$ or $YA = \left(\text{their } \lvert\overrightarrow{AX}\rvert\right)\tan\theta$, where $\theta$ is their acute or obtuse angle between $l_1$ and $l_2$ | M1 |
| | $YA = 55.71758... = 55.7$ (1 dp) | anything that rounds to 55.7 | A1 |
| | | | **[2]** |
| (e) **Way 1** | $\{A_{\lambda=2}, X_{\lambda=5} \Rightarrow \text{So } AX = 2AB \Rightarrow \text{So at } B,\ \lambda = 3.5 \text{ or } \lambda = 0.5)\}$ | | |
| | $\overrightarrow{OB} = \begin{pmatrix}4\\28\\4\end{pmatrix} + 3.5\begin{pmatrix}-1\\-5\\1\end{pmatrix}; = \begin{pmatrix}0.5\\10.5\\7.5\end{pmatrix}$ | Substitutes **either** $\lambda = \dfrac{(\text{their } \lambda_X \text{ found in } (a)) + 2}{2}$ **or** $\lambda_b = 3 - \dfrac{(\text{their } \lambda_X \text{ found in } (a))}{2}$ into $l_1$ | M1; |
| | $\overrightarrow{OB} = \begin{pmatrix}4\\28\\4\end{pmatrix} + 0.5\begin{pmatrix}-1\\-5\\1\end{pmatrix}; = \begin{pmatrix}3.5\\25.5\\4.5\end{pmatrix}$ | At least one position vector is correct. (Also allow coordinates). | A1 |
| | | Both position vectors are correct. (Also allow coordinates). | A1 |
| | | | **[3]** |
| | | | **13** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **6.** (e) **Way 2** | $\left\{AX = 2AB \Rightarrow AB = \frac{1}{2}AX.\text{ So, } \overrightarrow{OB} = \overrightarrow{OA} \pm \overrightarrow{AB} \Rightarrow \overrightarrow{OB} = \overrightarrow{OA} \pm \frac{1}{2}\overrightarrow{AX}\right\}$ | | |
| | $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} + 0.5\begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix}; = \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix}$ | Applies either $\overrightarrow{OA} + 0.5\overrightarrow{AX}$ **or** $\overrightarrow{OA} - 0.5\overrightarrow{AX}$ where (their $\overrightarrow{AX}$) $= \pm\left[(\text{their } \overrightarrow{OX}) - \overrightarrow{OA}\right]$ | M1; |
| | $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} - 0.5\begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix}; = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}$ | At least one position vector is correct (Also allow coordinates) | A1 |
| | | Both position vectors are correct (Also allow coordinates) | A1 |
| | | | **[3]** |
| **6.** (e) **Way 3** | $\overrightarrow{AB} = \begin{pmatrix} 4-\lambda \\ 28-5\lambda \\ 4+\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} = \begin{pmatrix} 2-\lambda \\ 10-5\lambda \\ -2+\lambda \end{pmatrix} = \begin{pmatrix} 1(2-\lambda) \\ 5(2-\lambda) \\ -1(2-\lambda) \end{pmatrix};\ \overrightarrow{AX} = \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix}$ $AX^2 = 243 \Rightarrow$ $AB^2 = 27(2-\lambda)^2$ <br> $AX = 2AB \Rightarrow AX^2 = 4AB^2 \Rightarrow 243 = 4(27)(2-\lambda)^2 \Rightarrow (2-\lambda)^2 = \frac{9}{4}$ **or** $27\lambda^2 - 108\lambda + \frac{189}{4} = 0$ <br> **or** $108\lambda^2 - 432\lambda + 189 = 0$ **or** $4\lambda^2 - 16\lambda + 7 = 0 \Rightarrow \lambda = 3.5$ or $\lambda = 0.5$ | | |
| | $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + 3.5\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}; = \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix}$ | Full method of solving for $\lambda$ the equation $AX^2 = 4AB^2$ using (their $\overrightarrow{AX}$) and $\overrightarrow{AB}$ **and** substitutes at least one of their values for $\lambda$ into $l_1$ | M1; |
| | $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 28 \\ 4 \end{pmatrix} + 0.5\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}; = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}$ | At least one position vector is correct (Also allow coordinates) | A1 |
| | | Both position vectors are correct (Also allow coordinates) | A1 |
| | **Note:** $AX = 2AB \Rightarrow \overrightarrow{AX} = \pm 2\overrightarrow{AB}$. Hence, $\lambda = 3.5$ or $\lambda = 0.5$ can be found from solving either $x: -3 = \pm 2(2-\lambda)$ or $y: -15 = \pm 2(10-5\lambda)$ or $z: -3 = \pm 2(-2+\lambda)$ | | **[3]** |
| **6.** (e) **Way 4** | $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} + 0.5\begin{pmatrix} 3 \\ 15 \\ -3 \end{pmatrix}; = \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix}$ | Applies **either** (their $\overrightarrow{OX}$) $+ 0.5\overrightarrow{XA}$ **or** (their $\overrightarrow{OX}$) $+ 1.5\overrightarrow{XA}$ where (their $\overrightarrow{XA}$) $= \overrightarrow{OA} -$ (their $\overrightarrow{OX}$) | M1; |
| | $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} + 1.5\begin{pmatrix} 3 \\ 15 \\ -3 \end{pmatrix}; = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}$ | At least one position vector is correct (Also allow coordinates) | A1 |
| | | Both position vectors are correct (Also allow coordinates) | A1 |
| | | | **[3]** |
| **6.** (e) **Way 5** | $\overrightarrow{OB} = 0.5\left(\begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} + \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix}\right); = \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix}$ | Applies $\frac{1}{2}\left[(\text{their } \overrightarrow{OX}) + \overrightarrow{OA}\right]$ | M1; |
| | $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} - 0.5\begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix}; = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}$ | At least one position vector is correct (Also allow coordinates) | A1 |
| | | Both position vectors are correct (Also allow coordinates) | A1 |
| | | | **[3]** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **6.** (e) **Way 6** | $\left\{\left\lvert\overrightarrow{AX}\right\rvert = 9\sqrt{3}\,, \lvert d_1\rvert = 3\sqrt{3} \Rightarrow K = \frac{9\sqrt{3}}{3\sqrt{3}} = 3 \Rightarrow \overrightarrow{AX} = 3\mathbf{d}_1;\text{ So, } \overrightarrow{OB} = \overrightarrow{OA} \pm \tfrac{1}{2}\overrightarrow{AX} = \overrightarrow{OA} \pm \tfrac{1}{2}(3\mathbf{d}_1)\right\}$ | | |
| | $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} + 0.5\left(3\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}\right); = \begin{pmatrix} 0.5 \\ 10.5 \\ 7.5 \end{pmatrix}$ | Applies either $\overrightarrow{OA} + 0.5(K\mathbf{d}_1)$ **or** $\overrightarrow{OA} - 0.5(K\mathbf{d}_1)$, where $K = \frac{\text{their } \lvert\overrightarrow{AX}\rvert}{3\sqrt{3}}$ | M1; |
| | $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} - 0.5\left(3\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}\right); = \begin{pmatrix} 3.5 \\ 25.5 \\ 4.5 \end{pmatrix}$ | At least one position vector is correct (Also allow coordinates) | A1 |
| | | Both position vectors are correct (Also allow coordinates) | A1 |
| | | | **[3]** |

| | **Question 6 Notes** | |
|---|---|---|
| **6.** (a) | **Note** | M1 can be implied by at least two correct follow through coordinates from their $\lambda$ or from their $\mu$ |
| (b) | **Note** | ***Evaluating*** the dot product (i.e. $(-1)(3) + (-5)(0) + (1)(-4)$) is not required for the M1, dM1 marks. |
| | **Note** | **For M1 dM1:** Allow one slip in writing down their direction vectors, $\mathbf{d}_1$ and $\mathbf{d}_2$ |
| | **Note** | Allow M1 dM1 for $\left(\sqrt{(-1)^2 + (-5)^2 + (1)^2}\,.\sqrt{(3)^2 + (0)^2 + (-4)^2}\right)\cos\theta = \pm\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$ |
| | **Note** | $\theta = 1.297995...^c$, (without evidence of awrt 74.37) is A0 |

| | | | |
|---|---|---|---|
| **6.** (b) **Way 2** | <u>**Alternative Method: Vector Cross Product**</u> | | |
| | **Only apply this scheme if it is clear that a vector cross product method is being applied.** | | |
| | $\mathbf{d}_1 \times \mathbf{d}_2 = \underline{\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}} = \left\{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -5 & 1 \\ 3 & 0 & -4 \end{vmatrix} = 20\mathbf{i} - \mathbf{j} + 15\mathbf{k}\right\}$ | Realisation that the vector cross product is required between $\mathbf{d}_1$ and $\mathbf{d}_2$ or a multiple of $\mathbf{d}_1$ and $\mathbf{d}_2$ | M1 |
| | $\sin\theta = \frac{\sqrt{(20)^2 + (-1)^2 + (15)^2}}{\sqrt{(-1)^2 + (-5)^2 + (1)^2}\,.\sqrt{(3)^2 + (0)^2 + (-4)^2}}$ | Applies the vector product formula between $\mathbf{d}_1$ and $\mathbf{d}_2$ or a multiple of $\mathbf{d}_1$ and $\mathbf{d}_2$ | dM1 |
| | $\sin\theta = \frac{\sqrt{626}}{\sqrt{27}\,.\sqrt{25}} \Rightarrow \theta = 74.36964117... = 74.37$ (2 dp) | awrt 74.37 seen in (b) only | A1 |
| | | | **[3]** |

| | | |
|---|---|---|
| **6.** (c) | **M1** | Finds the difference between their $\overrightarrow{OX}$ and $\overrightarrow{OA}$ and applies Pythagoras to the result to find $AX$ or $XA$ **OR** applies $\left\lvert(\text{their } \lambda_X \text{ found in } (a)) - 2\right\rvert.\sqrt{(-1)^2 + (-5)^2 + (1)^2}$ |
| | **Note** | For M1: Allow one slip in writing down their $\overrightarrow{OX}$ and $\overrightarrow{OA}$ |
| | **Note** | Allow M1A1 for $\begin{pmatrix} 3 \\ 15 \\ 3 \end{pmatrix}$ leading to $AX = \sqrt{(3)^2 + (15)^2 + (3)^2} = \sqrt{243} = 9\sqrt{3}$ |
| (e) | **Note** | Imply M1 for no working leading to any two components of one of the $\overrightarrow{OB}$ which are correct. |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **6.** (d) **Way 2** | $\dfrac{"9\sqrt{3}"}{YA} = \tan(90 - "74.36964...")$ | $\dfrac{\text{their } \lvert\overrightarrow{AX}\rvert}{YA} = \tan(90-\theta)$ or $AY = \dfrac{\text{their } \lvert\overrightarrow{AX}\rvert}{\tan(90-\theta)}$, where $\theta$ is the acute or obtuse angle between $l_1$ and $l_2$ | M1 |
| | $YA = 55.71758... = 55.7$ (1 dp) | anything that rounds to 55.7 | A1 |
| | | | **[2]** |
| **6.** (d) **Way 3** | $\dfrac{YA}{\sin("74.36964...")} = \dfrac{"9\sqrt{3}"}{\sin(90 - "74.36964...")}$ | $\dfrac{YA}{\sin\theta} = \dfrac{\text{their } \lvert\overrightarrow{AX}\rvert}{\sin(90-\theta)}$ o.e., where $\theta$ is the acute or obtuse angle between $l_1$ and $l_2$ | M1 |
| | $YA = \dfrac{9\sqrt{3}\sin(74.36964...)}{\sin(15.63036...)} = 55.71758... = 55.7$ (1 dp) | anything that rounds to 55.7 | A1 |
| | | | **[2]** |
| **6.** (d) **Way 4** | $\mathbf{d}_1 = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}$, $\overrightarrow{OY} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 5+3\mu \\ 3 \\ 1-4\mu \end{pmatrix}$ | | |
| | $\overrightarrow{YA} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} - \begin{pmatrix} 5+3\mu \\ 3 \\ 1-4\mu \end{pmatrix} = \begin{pmatrix} -3-3\mu \\ 15 \\ 5+4\mu \end{pmatrix}$ <br> $\overrightarrow{YA}\bullet\mathbf{d}_1 = 0 \Rightarrow \begin{pmatrix} -3-3\mu \\ 15 \\ 5+4\mu \end{pmatrix}\bullet\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} = 0$ <br> $\Rightarrow 3+3\mu-75+5+4\mu=0 \Rightarrow \mu=\dfrac{67}{7}$ <br> $YA^2 = \left(-3-3\left(\dfrac{67}{7}\right)\right)^2 + (15)^2 + \left(5+4\left(\dfrac{67}{7}\right)\right)^2$ | (Allow a sign slip in copying $\mathbf{d}_1$) <br> Applies $\overrightarrow{YA}\bullet\mathbf{d}_1 = 0$ or $\overrightarrow{AY}\bullet\mathbf{d}_1 = 0$ or $\overrightarrow{YA}\bullet(K\mathbf{d}_1) = 0$ or $\overrightarrow{AY}\bullet(K\mathbf{d}_1) = 0$ to find $\mu$ and applies Pythagoras to find a numerical expression for $AY^2$ or for the distance $AY$ | M1 |
| | So, $YA = \sqrt{\left(-\dfrac{222}{7}\right)^2 + (15)^2 + \left(\dfrac{303}{7}\right)^2}$ | | |
| | $= 55.71758... = 55.7$ (1 dp) | anything that rounds to 55.7 | A1 |
| | **Note:** $\overrightarrow{OY} = \dfrac{236}{7}\mathbf{i} + 3\mathbf{j} - \dfrac{261}{7}\mathbf{k}$, $\overrightarrow{AY} = -\dfrac{222}{7}\mathbf{i} + 15\mathbf{j} + \dfrac{303}{7}\mathbf{k}$ | | **[2]** |

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**7.**

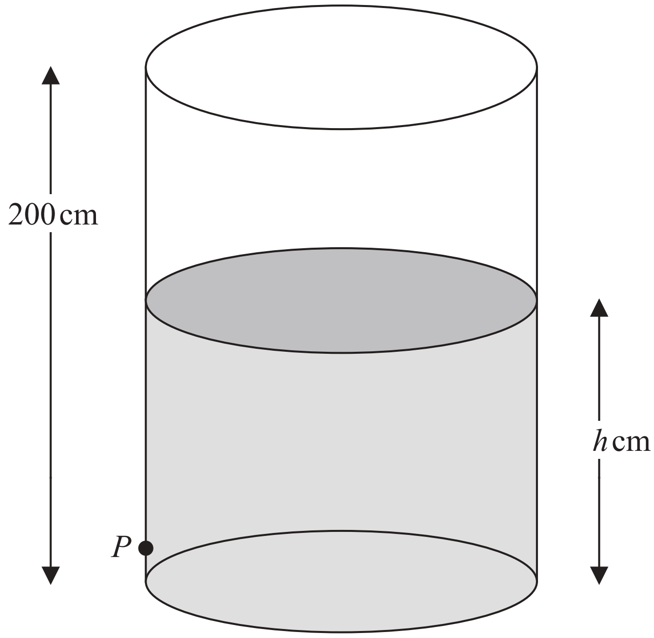


**Figure 3**

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole $P$ on the side of the tank.

At time $t$ minutes after the leaking starts, the height of water in the tank is $h$ cm.

The height $h$ cm of the water in the tank satisfies the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where $k$ is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of $k$.

**(2)**

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of $k$, to find the value of $t$ when $h = 50$

**(6)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **7.** | $\frac{\mathrm{d}h}{\mathrm{d}t} = k\sqrt{(h-9)},\ \ 9 < h \leq 200;\quad h = 130,\ \frac{\mathrm{d}h}{\mathrm{d}t} = -1.1$ | | |
| (a) | $-1.1 = k\sqrt{(130-9)} \Rightarrow k = ...$ | Substitutes $h = 130$ and either $\frac{\mathrm{d}h}{\mathrm{d}t} = -1.1$ or $\frac{\mathrm{d}h}{\mathrm{d}t} = 1.1$ into the printed equation and rearranges to give $k = ...$ | M1 |
| | so, $k = -\frac{1}{10}$ or $-0.1$ | $k = -\frac{1}{10}$ or $-0.1$ | A1 |
| | | | **[2]** |
| (b) **Way 1** | $\int \frac{\mathrm{d}h}{\sqrt{(h-9)}} = \int k\,\mathrm{d}t$ | Separates the variables correctly. $\mathrm{d}h$ and $\mathrm{d}t$ should not be in the wrong positions, although this mark can be implied by later working. Ignore the integral signs. | B1 |
| | $\int (h-9)^{-\frac{1}{2}}\,\mathrm{d}h = \int k\,\mathrm{d}t$ | | |
| | $\frac{(h-9)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = kt\ (+c)$ | Integrates $\frac{\pm\lambda}{\sqrt{(h-9)}}$ to give $\pm\mu\sqrt{(h-9)}\,;\ \lambda, \mu \neq 0$ | M1 |
| | | $\frac{(h-9)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = kt$ or $\frac{(h-9)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = (\text{their } k)t$, with/without $+c$, or equivalent, which can be un-simplified or simplified. | A1 |
| | $\{t = 0, h = 200 \Rightarrow\}\ 2\sqrt{(200-9)} = k(0) + c$ | Some evidence of applying both $t = 0$ and $h = 200$ to changed equation containing a constant of integration, e.g. $c$ or $A$ | M1 |
| | $\Rightarrow c = 2\sqrt{191} \Rightarrow 2(h-9)^{\frac{1}{2}} = -0.1t + 2\sqrt{191}$<br>$\{h = 50 \Rightarrow\}\ 2\sqrt{(50-9)} = -0.1t + 2\sqrt{191}$<br>$t = ...$ | **dependent on the previous M mark** Applies $h = 50$ and their value of $c$ to their changed equation and rearranges to find the value of $t = ...$ | dM1 |
| | $t = 20\sqrt{191} - 20\sqrt{41}$<br>or $t = 148.3430145... = 148$ (minutes) (nearest minute) | $t = 20\sqrt{191} - 20\sqrt{41}$ isw or awrt 148 | A1 cso |
| | | | **[6]** |
| (b) **Way 2** | $\int_{200}^{50} \frac{\mathrm{d}h}{\sqrt{(h-9)}} = \int_0^T k\,\mathrm{d}t$ | Separates the variables correctly. $\mathrm{d}h$ and $\mathrm{d}t$ should not be in the wrong positions, although this mark can be implied by later working. Integral signs and limits not necessary. | B1 |
| | $\int_{200}^{50} (h-9)^{-\frac{1}{2}}\,\mathrm{d}h = \int_0^T k\,\mathrm{d}t$ | | |
| | $\left[\frac{(h-9)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_{200}^{50} = \left[kt\right]_0^T$ | Integrates $\frac{\pm\lambda}{\sqrt{(h-9)}}$ to give $\pm\mu\sqrt{(h-9)}\,;\ \lambda, \mu \neq 0$ | M1 |
| | | $\frac{(h-9)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = kt$ or $\frac{(h-9)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} = (\text{their } k)t$, with/without limits, or equivalent, which can be un-simplified or simplified. | A1 |
| | $2\sqrt{41} - 2\sqrt{191} = kt$ or $kT$ | Attempts to apply limits of $h = 200, h = 50$ and (can be implied) $t = 0$ to their changed equation | M1 |
| | $t = \frac{2\sqrt{41} - 2\sqrt{191}}{-0.1}$ | **dependent on the previous M mark** Then rearranges to find the value of $t = ...$ | dM1 |
| | $t = 20\sqrt{191} - 20\sqrt{41}$<br>or $t = 148.3430145... = 148$ (minutes) (nearest minute) | $t = 20\sqrt{191} - 20\sqrt{41}$ **or** awrt 148 **or** 2 hours and awrt 28 minutes | A1 cso |
| | | | **[6]** |
| | | | **8** |

| | | |
|---|---|---|
| | | **Question 7 Notes** |
| **7.** (b) | **Note** | Allow first B1 for writing $\dfrac{\mathrm{d}t}{\mathrm{d}h} = \dfrac{1}{k\sqrt{(h-9)}}$ or $\dfrac{\mathrm{d}t}{\mathrm{d}h} = \dfrac{1}{(\text{their } k)\sqrt{(h-9)}}$ or equivalent |
| | **Note** | $\dfrac{\mathrm{d}t}{\mathrm{d}h} = \dfrac{1}{k\sqrt{(h-9)}}$ leading to $t = \dfrac{2}{k}\sqrt{(h-9)}\ (+c)$ with/without $+c$ is B1M1A1 |
| | **Note** | After finding $k = 0.1$ in part (a), it is only possible to gain full marks in part (b) by **initially writing** $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -k\sqrt{(h-9)}$ **or** $\displaystyle\int \frac{\mathrm{d}h}{\sqrt{(h-9)}} = \int k\,\mathrm{d}t$ **or** $\dfrac{\mathrm{d}h}{\mathrm{d}t} = -0.1\sqrt{(h-9)}$ **or** $\displaystyle\int \frac{\mathrm{d}h}{\sqrt{(h-9)}} = \int 0.1\,\mathrm{d}t$<br>Otherwise, those candidates who find $k = 0.1$ in part (a), should lose at least the final A1 mark in part (b). |

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**8.**

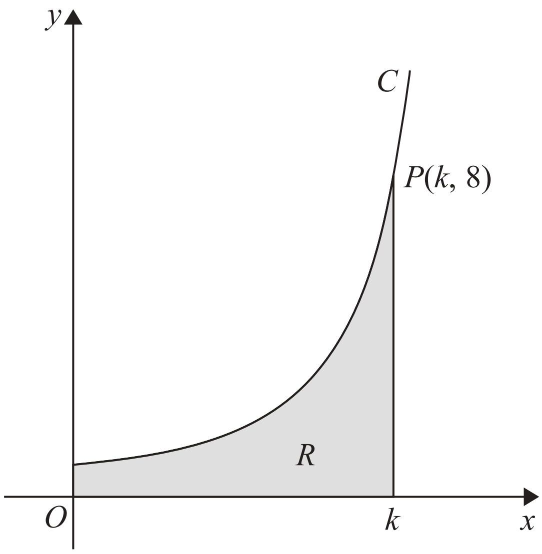


Diagram not drawn to scale

**Figure 4**

Figure 4 shows a sketch of part of the curve $C$ with parametric equations

$$x = 3\theta\sin\theta, \qquad y = \sec^3\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point $P(k, 8)$ lies on $C$, where $k$ is a constant.

(a) Find the exact value of $k$.

**(2)**

The finite region $R$, shown shaded in Figure 4, is bounded by the curve $C$, the $y$-axis, the $x$-axis and the line with equation $x = k$.

(b) Show that the area of $R$ can be expressed in the form

$$\lambda\int_{\alpha}^{\beta}\left(\theta\sec^2\theta + \tan\theta\sec^2\theta\right)\mathrm{d}\theta$$

where $\lambda$, $\alpha$ and $\beta$ are constants to be determined.

**(4)**

(c) Hence use integration to find the exact value of the area of $R$.

**(6)**

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **8.** | $x = 3\theta\sin\theta,\ \ y = \sec^3\theta,\ \ 0 \le \theta < \frac{\pi}{2}$ | | |
| (a) | $\{\text{When } y=8,\}\ 8 = \sec^3\theta \Rightarrow \cos^3\theta = \frac{1}{8} \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$<br>$k \text{ (or } x) = 3\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)$ | Sets $y=8$ to find $\theta$ **and** attempts to substitute their $\theta$ into $x = 3\theta\sin\theta$ | M1 |
| | so $k \text{ (or } x) = \frac{\sqrt{3}\pi}{2}$ | $\frac{\sqrt{3}\pi}{2}$ or $\frac{3\pi}{2\sqrt{3}}$ | A1 |
| | **Note:** Obtaining two value for $k$ without accepting the correct value is final A0 | | **[2]** |
| (b) | $\frac{dx}{d\theta} = 3\sin\theta + 3\theta\cos\theta$ | $3\theta\sin\theta \to 3\sin\theta + 3\theta\cos\theta$<br>Can be implied by later working | B1 |
| | $\left\{\int y\frac{dx}{d\theta}\{d\theta\}\right\} = \int(\sec^3\theta)(3\sin\theta + 3\theta\cos\theta)\{d\theta\}$ | Applies $\left(\pm K\sec^3\theta\right)\left(\text{their } \frac{dx}{d\theta}\right)$<br>Ignore integral sign and $d\theta$; $K \ne 0$ | M1 |
| | $= 3\int \theta\sec^2\theta + \tan\theta\sec^2\theta\, d\theta$ | Achieves the correct result no errors in their working, e.g. bracketing or manipulation errors.<br>**Must have** integral sign and $d\theta$ in their final answer. | A1 * |
| | $x=0$ and $x=k \Rightarrow \underline{\alpha=0}$ and $\underline{\beta=\frac{\pi}{3}}$ | $\alpha=0$ and $\beta=\frac{\pi}{3}$ or evidence of $0\to0$ and $k\to\frac{\pi}{3}$ | B1 |
| | **Note:** The work for the final B1 mark must be seen in part (b) only. | | **[4]** |
| (c)<br>**Way 1** | $\left\{\int\theta\sec^2\theta\, d\theta\right\} = \theta\tan\theta - \int\tan\theta\{d\theta\}$ | $\theta\sec^2\theta \to A\theta g(\theta) - B\int g(\theta),\ A>0,\ B>0,$<br>where $g(\theta)$ is a trigonometric function in $\theta$ **and** $g(\theta) =$ their $\int\sec^2\theta d\theta$. [Note: $g(\theta) \ne \sec^2\theta$] | M1 |
| | | **dependent on the previous M mark**<br>**Either** $\lambda\theta\sec^2\theta \to A\theta\tan\theta - B\int\tan\theta,\ A>0,\ B>0$<br>**or** $\theta\sec^2\theta \to \theta\tan\theta - \int\tan\theta$ | dM1 |
| | $= \theta\tan\theta - \ln(\sec\theta)$<br>**or** $= \theta\tan\theta + \ln(\cos\theta)$ | $\theta\sec^2\theta \to \theta\tan\theta - \ln(\sec\theta)$ or $\theta\tan\theta + \ln(\cos\theta)$ **or**<br>$\lambda\theta\sec^2\theta \to \lambda\theta\tan\theta - \lambda\ln(\sec\theta)$ or $\lambda\theta\tan\theta + \lambda\ln(\cos\theta)$ | A1 |
| | **Note: Condone** $\theta\sec^2\theta \to \theta\tan\theta - \ln(\sec x)$ or $\theta\tan\theta + \ln(\cos x)$ for A1 | | |
| | $\left\{\int\tan\theta\sec^2\theta\, d\theta\right\}$ | $\tan\theta\sec^2\theta$ or $\lambda\tan\theta\sec^2\theta \to \pm C\tan^2\theta$ or $\pm C\sec^2\theta$<br>or $\pm Cu^{-2}$, where $u=\cos\theta$ | M1 |
| | $= \frac{1}{2}\tan^2\theta$ or $\frac{1}{2}\sec^2\theta$<br>or $\frac{1}{2u^2}$ where $u=\cos\theta$<br>or $\frac{1}{2}u^2$ where $u=\tan\theta$ | $\tan\theta\sec^2\theta \to \frac{1}{2}\tan^2\theta$ or $\frac{1}{2}\sec^2\theta$ or $\frac{1}{2\cos^2\theta}$ or $\tan^2\theta - \frac{1}{2}\sec^2\theta$<br>or $0.5u^{-2}$, where $u=\cos\theta$ or $0.5u^2$, where $u=\tan\theta$<br>or $\lambda\tan\theta\sec^2\theta \to \frac{\lambda}{2}\tan^2\theta$ or $\frac{\lambda}{2}\sec^2\theta$ or $\frac{\lambda}{2\cos^2\theta}$<br>or $0.5\lambda u^{-2}$, where $u=\cos\theta$ or $0.5\lambda u^2$, where $u=\tan\theta$ | A1 |
| | $\{\text{Area}(R)\} = \left[3\theta\tan\theta - 3\ln(\sec\theta) + \frac{3}{2}\tan^2\theta\right]_0^{\frac{\pi}{3}}$ **or** $\left[3\theta\tan\theta - 3\ln(\sec\theta) + \frac{3}{2}\sec^2\theta\right]_0^{\frac{\pi}{3}}$ | | |
| | $= \left(3\left(\frac{\pi}{3}\right)\sqrt{3} - 3\ln 2 + \frac{3}{2}(3)\right) - (0)$ **or** $\left(3\left(\frac{\pi}{3}\right)\sqrt{3} - 3\ln 2 + \frac{3}{2}(4)\right) - \left(\frac{3}{2}\right)$ | | |
| | $= \frac{9}{2} + \sqrt{3}\pi - 3\ln 2$ **or** $\frac{9}{2} + \sqrt{3}\pi + 3\ln\left(\frac{1}{2}\right)$ **or** $\frac{9}{2} + \sqrt{3}\pi - \ln 8$ **or** $\ln\left(\frac{1}{8}e^{\frac{9}{2}+\sqrt{3}\pi}\right)$ | | A1 o.e. |
| | | | **[6]** |
| | | | **12** |

| Question Number | Scheme | Notes | Marks |
|---|---|---|---|
| **8.** (c) **Way 2** | **<u>Way 2 for the first 5 marks:</u>** Applying integration by parts on $\int(\theta + \tan\theta)\sec^2\theta\,\mathrm{d}\theta$ | | |
| | $\int(\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta = \int(\theta + \tan\theta)\sec^2\theta\,\mathrm{d}\theta,$ | $\left\{\begin{array}{l} u = \theta + \tan\theta \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}\theta} = 1 + \sec^2\theta \\ \dfrac{\mathrm{d}v}{\mathrm{d}\theta} = \sec^2\theta \Rightarrow v = \tan\theta = \mathrm{g}(\theta) \end{array}\right\}$ | |
| | $\mathrm{h}(\theta)$ and $\mathrm{g}(\theta)$ are trigonometric functions in $\theta$ **and** $\mathrm{g}(\theta) =$ their $\int\sec^2\theta\,\mathrm{d}\theta$. [**Note:** $\mathrm{g}(\theta) \neq \sec^2\theta$] | | |
| | | $A(\theta + \tan\theta)\mathrm{g}(\theta) - B\int(1 + \mathrm{h}(\theta))\mathrm{g}(\theta),\ A > 0, B > 0$ | M1 |
| | $= (\theta + \tan\theta)\tan\theta - \int(1 + \sec^2\theta)\tan\theta\{\mathrm{d}\theta\}$ | **dependent on the previous M mark** **Either** $\pm\left[(\theta + \tan\theta)\sec^2\theta\right] \to$ $A(\theta + \tan\theta)\tan\theta - B\int(1 + \mathrm{h}(\theta))\tan\theta,\ A \neq 0, B > 0$ **or** $(\theta + \tan\theta)\tan\theta - \int(1 + \mathrm{h}(\theta))\tan\theta$ | dM1 |
| | $= (\theta + \tan\theta)\tan\theta - \int(\tan\theta + \tan\theta\sec^2\theta)\{\mathrm{d}\theta\}$ | | |
| | $= (\theta + \tan\theta)\tan\theta - \ln(\sec\theta) - \int\tan\theta\sec^2\theta\{\mathrm{d}\theta\}$ | $(\theta + \tan\theta)\tan\theta - \ln(\sec\theta)$ o.e. or $\pm\left[(\theta + \tan\theta)\tan\theta - \ln(\sec\theta)\right]$ o.e. | A1 |
| | $= (\theta + \tan\theta)\tan\theta - \ln(\sec\theta) - \frac{1}{2}\tan^2\theta$ or $= (\theta + \tan\theta)\tan\theta - \ln(\sec\theta) - \frac{1}{2}\sec^2\theta$ etc. | $\tan\theta\sec^2\theta \to \pm C\tan^2\theta$ or $\pm C\sec^2\theta$ | M1 |
| | | $(\theta + \tan\theta)\tan\theta - \frac{1}{2}\tan^2\theta$ or $(\theta + \tan\theta)\tan\theta - \frac{1}{2}\sec^2\theta$ | A1 |
| | **Note** | Allow the first two marks in part (c) for $\theta\tan\theta - \int\tan\theta$ embedded in their working | |
| | **Note** | Allow the first three marks in part (c) for $\theta\tan\theta - \ln(\sec\theta)$ embedded in their working | |
| | **Note** | Allow 3rd M1 2nd A1 marks for either $\tan^2\theta - \frac{1}{2}\tan^2\theta$ or $\tan^2\theta - \frac{1}{2}\sec^2\theta$ embedded in their working | |
| | **Question 8 Notes** | | |
| **8.** (a) | **Note** | Allow M1 for an answer of $k =$ awrt 2.72 without reference to $\dfrac{\sqrt{3}\pi}{2}$ or $\dfrac{3\pi}{2\sqrt{3}}$ | |
| | **Note** | Allow M1 for an answer of $k = 3\left(\arccos(\tfrac{1}{2})\right)\sin\left(\arccos(\tfrac{1}{2})\right)$ without reference to $\dfrac{\sqrt{3}\pi}{2}$ or $\dfrac{3\pi}{2\sqrt{3}}$ | |
| | **Note** | E.g. allow M1 for $\theta = 60°$, leading to $k = 3(60)\sin(60)$ or $k = 90\sqrt{3}$ | |

**Question 8 Notes Continued**

| | | |
|---|---|---|
| **8.** (b) | **Note** | To gain A1, $\mathrm{d}\theta$ does not need to appear until they obtain $3\int(\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta$ |
| | **Note** | For M1, their $\frac{\mathrm{d}x}{\mathrm{d}\theta}$, where their $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 3\theta\sin\theta$, needs to be a trigonometric function in $\theta$ |
| | **Note** | Writing $\int(\sec^3\theta)(3\sin\theta + 3\theta\cos\theta) = 3\int(\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta$ is sufficient for B1M1A1 |
| | **Note** | Writing $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin\theta + 3\theta\cos\theta$ followed by writing $\int y\frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta = 3\int(\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta$ is sufficient for B1M1A1 |
| | **Note** | The final A mark would be lost for $\int\frac{1}{\cos^3\theta}3\sin\theta + 3\theta\cos\theta = 3\int(\theta\sec^2\theta + \tan\theta\sec^2\theta)\,\mathrm{d}\theta$ [lack of brackets in this particular case]. |
| | **Note** | Give 2nd B0 for $\alpha = 0$ and $\beta = 60°$, without reference to $\beta = \frac{\pi}{3}$ |
| (c) | **Note** | A decimal answer of 7.861956551... (without a correct **exact** answer) is A0. |
| | **Note** | First three marks are for integrating $\theta\sec^2\theta$ with respect to $\theta$ |
| | **Note** | Fourth and fifth marks are for integrating $\tan\theta\sec^2\theta$ with respect to $\theta$ |
| | **Note** | Candidates are not penalised for writing $\ln\lvert\sec\theta\rvert$ as either $\ln(\sec\theta)$ or $\ln\sec\theta$ |
| | **Note** | $\theta\sec^2\theta \to \theta\tan\theta + \ln(\sec\theta)$ **WITH NO INTERMEDIATE WORKING** is M0M0A0 |
| | **Note** | $\theta\sec^2\theta \to \theta\tan\theta - \ln(\cos\theta)$ **WITH NO INTERMEDIATE WORKING** is M0M0A0 |
| | **Note** | $\theta\sec^2\theta \to \theta\tan\theta - \ln(\sec\theta)$ **WITH NO INTERMEDIATE WORKING** is M1M1A1 |
| | **Note** | $\theta\sec^2\theta \to \theta\tan\theta + \ln(\cos\theta)$ **WITH NO INTERMEDIATE WORKING** is M1M1A1 |
| | **Note** | Writing a correct $uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}$ with $u = \theta$, $\frac{\mathrm{d}v}{\mathrm{d}\theta} = \tan\theta$, $\frac{\mathrm{d}u}{\mathrm{d}\theta} = 1$ and $v$ = their $g(\theta)$ and making one error in the direct application of this formula is 1st M1 only. |

| | | | |
|---|---|---|---|
| **8.** (c) | Alternative method for finding $\int\tan\theta\sec^2\theta\,\mathrm{d}\theta$ | | |
| | $\left\{\begin{array}{ll} u = \tan\theta & \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = \sec^2\theta \\ \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sec^2\theta & \Rightarrow v = \tan\theta \end{array}\right\}$ | | |
| | $\int\tan\theta\sec^2\theta\,\mathrm{d}\theta = \tan^2\theta - \int\tan\theta\sec^2\theta\,\mathrm{d}\theta$<br>$\Rightarrow 2\int\tan\theta\sec^2\theta\,\mathrm{d}\theta = \tan^2\theta$ | | |
| | $\int\tan\theta\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\tan^2\theta$ | $\tan\theta\sec^2\theta$ or $\to \pm C\tan^2\theta$ | M1 |
| | | $\tan\theta\sec^2\theta \to \frac{1}{2}\tan^2\theta$ | A1 |
| | **or** $\left\{\begin{array}{ll} u = \sec\theta & \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = \sec\theta\tan\theta \\ \frac{\mathrm{d}v}{\mathrm{d}\theta} = \sec\theta\tan\theta & \Rightarrow v = \sec\theta \end{array}\right\}$ | | |
| | $\Rightarrow \int\tan\theta\sec^2\theta\,\mathrm{d}\theta = \sec^2\theta - \int\sec^2\theta\tan\theta\,\mathrm{d}\theta$<br>$\Rightarrow 2\int\tan\theta\sec^2\theta\,\mathrm{d}\theta = \sec^2\theta$ | | |
| | $\int\tan\theta\sec^2\theta\,\mathrm{d}\theta = \frac{1}{2}\sec^2\theta$ | $\tan\theta\sec^2\theta$ or $\to \pm C\sec^2\theta$ | M1 |
| | | $\tan\theta\sec^2\theta \to \frac{1}{2}\sec^2\theta$ | A1 |