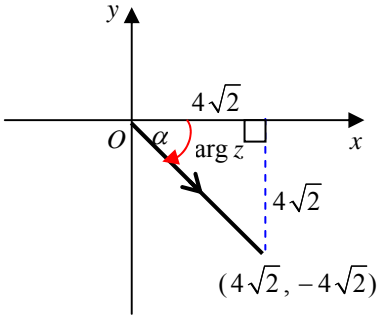


June 2009
6668 Further Pure Mathematics FP2 (new)
Mark Scheme

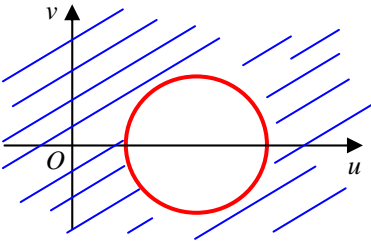
Question Number	Scheme	Marks
Q1 (a)	$\frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$	$\frac{1}{2r} - \frac{1}{2(r+2)}$ B1 aef (1)
Q1 (b)	$\sum_{r=1}^n \frac{4}{r(r+2)} = \sum_{r=1}^n \left(\frac{2}{r} - \frac{2}{r+2} \right)$ $= \left(\frac{2}{1} - \frac{2}{3} \right) + \left(\frac{2}{2} - \frac{2}{4} \right) + \dots$ $\dots\dots\dots + \left(\frac{2}{n-1} - \frac{2}{n+1} \right) + \left(\frac{2}{n} - \frac{2}{n+2} \right)$ $= \frac{2}{1} + \frac{2}{2}; - \frac{2}{n+1} - \frac{2}{n+2}$ $= 3 - \frac{2}{n+1} - \frac{2}{n+2}$ $= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)}$ $= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{(n+1)(n+2)}$ $= \frac{3n^2 + 5n}{(n+1)(n+2)}$ $= \frac{n(3n+5)}{(n+1)(n+2)}$	<p>List the first two terms and the last two terms</p> <p>M1</p> <p>Includes the first two underlined terms and includes the final two underlined terms.</p> <p>M1</p> <p>A1</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <p>Attempt to combine to an at least 3 term fraction to a single fraction and an attempt to take out the brackets from their numerator.</p> </div> <p>M1</p> <p>Correct Result</p> <p>A1 cso AG</p> <p>(5)</p> <p>[6]</p>

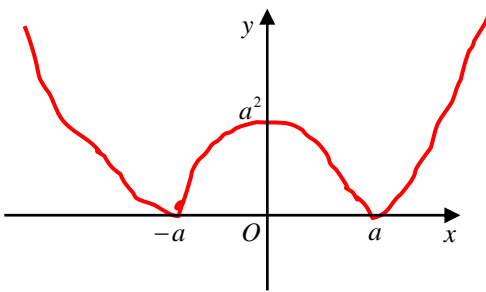
Question Number	Scheme	Marks
<p>Q2 (a)</p>	<p>$z^3 = 4\sqrt{2} - 4\sqrt{2}i$, $-\pi < \theta \leq \pi$</p>  <p>$r = \sqrt{(4\sqrt{2})^2 + (-4\sqrt{2})^2} = \sqrt{32 + 32} = \sqrt{64} = 8$</p> <p>$\theta = -\tan^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = -\frac{\pi}{4}$</p> <p>$z^3 = 8\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$</p> <p>So, $z = (8)^{\frac{1}{3}}\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$</p> <p>$\Rightarrow z = 2\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$</p> <p>Also, $z^3 = 8\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$ or $z^3 = 8\left(\cos\left(-\frac{9\pi}{4}\right) + i\sin\left(-\frac{9\pi}{4}\right)\right)$</p> <p>$\Rightarrow z = 2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ and $z = 2\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$</p> <p>Special Case 1: Award SC: M1M1A1M1A0A0 for ALL three of $2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$, $2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ and $2\left(\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$.</p> <p>Special Case 2: If r is incorrect (and not equal to 8) and candidate states the brackets () correctly then give the first accuracy mark ONLY where this is applicable.</p>	<p>A valid attempt to find the modulus and argument of $4\sqrt{2} - 4\sqrt{2}i$. M1</p> <p>Taking the cube root of the modulus and dividing the argument by 3. M1</p> <p>$2\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$ A1</p> <p>Adding or subtracting 2π to the argument for z^3 in order to find other roots. M1</p> <p>Any one of the final two roots A1</p> <p>Both of the final two roots. A1</p> <p>[6]</p>

Question Number	Scheme	Marks
Q3	$\sin x \frac{dy}{dx} - y \cos x = \sin 2x \sin x$ $\frac{dy}{dx} - \frac{y \cos x}{\sin x} = \frac{\sin 2x \sin x}{\sin x}$ $\frac{dy}{dx} - \frac{y \cos x}{\sin x} = \sin 2x$ <p>Integrating factor = $e^{\int -\frac{\cos x}{\sin x} dx} = e^{-\ln \sin x}$</p> $= \frac{1}{\sin x}$ $\left(\frac{1}{\sin x}\right) \frac{dy}{dx} - \frac{y \cos x}{\sin^2 x} = \frac{\sin 2x}{\sin x}$ $\frac{d}{dx} \left(\frac{y}{\sin x}\right) = \sin 2x \times \frac{1}{\sin x}$ $\frac{d}{dx} \left(\frac{y}{\sin x}\right) = 2 \cos x$ $\frac{y}{\sin x} = \int 2 \cos x dx$ $\frac{y}{\sin x} = 2 \sin x + K$ $y = 2 \sin^2 x + K \sin x$	<p>An attempt to divide every term in the differential equation by $\sin x$. Can be implied.</p> <p>M1</p> <p>dM1 A1 aef</p> <p>A1 aef</p> <p>M1</p> <p>A1</p> <p>dddM1</p> <p>A1 cao</p> <p>[8]</p>

Question Number	Scheme	Marks
Q4	$A = \frac{1}{2} \int_0^{2\pi} (a + 3 \cos \theta)^2 d\theta$ $(a + 3 \cos \theta)^2 = a^2 + 6a \cos \theta + 9 \cos^2 \theta$ $= a^2 + 6a \cos \theta + 9 \left(\frac{1 + \cos 2\theta}{2} \right)$ $A = \frac{1}{2} \int_0^{2\pi} \left(a^2 + 6a \cos \theta + \frac{9}{2} + \frac{9}{2} \cos 2\theta \right) d\theta$ $= \left(\frac{1}{2} \right) \left[a^2 \theta + 6a \sin \theta + \frac{9}{2} \theta + \frac{9}{4} \sin 2\theta \right]_0^{2\pi}$ $= \frac{1}{2} \left[(2\pi a^2 + 0 + 9\pi + 0) - (0) \right]$ $= \pi a^2 + \frac{9\pi}{2}$ <p>Hence, $\pi a^2 + \frac{9\pi}{2} = \frac{107}{2} \pi$</p> $a^2 + \frac{9}{2} = \frac{107}{2}$ $a^2 = 49$ <p>As $a > 0$, $a = 7$</p> <p>Some candidates may achieve $a = 7$ from incorrect working. Such candidates will not get full marks</p>	<p>Applies $\frac{1}{2} \int_0^{2\pi} r^2 (d\theta)$ with correct limits. Ignore $d\theta$.</p> <p>$\cos^2 \theta = \frac{\pm 1 \pm \cos 2\theta}{2}$</p> <p><u>Correct underlined expression.</u></p> <p>Integrated expression with at least 3 out of 4 terms of the form $\pm A\theta \pm B \sin \theta \pm C\theta \pm D \sin 2\theta$. Ignore the $\frac{1}{2}$. Ignore limits. $a^2 \theta + 6a \sin \theta +$ correct ft integration. Ignore the $\frac{1}{2}$. Ignore limits.</p> <p>Integrated expression equal to $\frac{107}{2} \pi$.</p> <p>$\pi a^2 + \frac{9\pi}{2}$</p> <p>$a = 7$</p> <p>[8]</p>

Question Number	Scheme	Marks
<p>Q5</p> <p>(a)</p> $y = \sec^2 x = (\sec x)^2$ $\frac{dy}{dx} = 2(\sec x)^1(\sec x \tan x) = 2\sec^2 x \tan x$ <p>Apply product rule:</p> $\left\{ \begin{array}{l} u = 2\sec^2 x \\ \frac{du}{dx} = 4\sec^2 x \tan x \end{array} \right. \quad \left\{ \begin{array}{l} v = \tan x \\ \frac{dv}{dx} = \sec^2 x \end{array} \right.$ $\frac{d^2y}{dx^2} = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$ <p>Hence, $\frac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$</p> <p>(b)</p> $y_{\frac{\pi}{4}} = (\sqrt{2})^2 = 2, \quad \left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = 2(\sqrt{2})^2(1) = 4$ $\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 6(\sqrt{2})^4 - 4(\sqrt{2})^2 = 24 - 8 = 16$ $\frac{d^3y}{dx^3} = 24\sec^3 x(\sec x \tan x) - 8\sec x(\sec x \tan x)$ $= 24\sec^4 x \tan x - 8\sec^2 x \tan x$ $\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 24(\sqrt{2})^4(1) - 8(\sqrt{2})^2(1) = 96 - 16 = 80$ $\sec x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + \frac{16}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{80}{6}\left(x - \frac{\pi}{4}\right)^3 + \dots$ $\left\{ \sec x \approx 2 + 4\left(x - \frac{\pi}{4}\right) + 8\left(x - \frac{\pi}{4}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{4}\right)^3 + \dots \right\}$	<p>Either $2(\sec x)^1(\sec x \tan x)$ or $2\sec^2 x \tan x$</p> <p>Two terms added with one of either $A \sec^2 x \tan^2 x$ or $B \sec^4 x$ in the correct form. Correct differentiation</p> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <p>Applies $\tan^2 x = \sec^2 x - 1$ leading to the correct result.</p> </div> <p>Both $y_{\frac{\pi}{4}} = 2$ and $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = 4$</p> <p>Attempts to substitute $x = \frac{\pi}{4}$ into both terms in the expression for $\frac{d^2y}{dx^2}$.</p> <p>Two terms differentiated with either $24\sec^4 x \tan x$ or $-8\sec^2 x \tan x$ being correct</p> $\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = 80$ <p>Applies a Taylor expansion with at least 3 out of 4 terms fit correctly. Correct Taylor series expansion.</p>	<p>B1 aef</p> <p>M1 A1</p> <p>A1 AG</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>B1</p> <p>M1 A1</p> <p>(4)</p> <p>(6)</p> <p>[10]</p>

Question Number	Scheme	Marks
<p>Q6</p> <p>(a)</p> <p>(b)</p>	$w = \frac{z}{z+i}, z = -i$ $w(z+i) = z \Rightarrow wz + iw = z \Rightarrow iw = z - wz$ $\Rightarrow iw = z(1-w) \Rightarrow z = \frac{iw}{(1-w)}$ $ z = 3 \Rightarrow \left \frac{iw}{1-w} \right = 3$ $\left\{ \begin{array}{l} iw = 3 1-w \Rightarrow w = 3 w-1 \Rightarrow w ^2 = 9 w-1 ^2 \\ \Rightarrow u+iv ^2 = 9 u+iv-1 ^2 \end{array} \right\}$ $\Rightarrow u^2 + v^2 = 9[(u-1)^2 + v^2]$ $\left\{ \begin{array}{l} \Rightarrow u^2 + v^2 = 9u^2 - 18u + 9 + 9v^2 \\ \Rightarrow 0 = 8u^2 - 18u + 8v^2 + 9 \end{array} \right\}$ $\Rightarrow 0 = u^2 - \frac{9}{4}u + v^2 + \frac{9}{8}$ $\Rightarrow \left(u - \frac{9}{8}\right)^2 - \frac{81}{64} + v^2 + \frac{9}{8} = 0$ $\Rightarrow \left(u - \frac{9}{8}\right)^2 + v^2 = \frac{9}{64}$ <p>{Circle} centre $\left(\frac{9}{8}, 0\right)$, radius $\frac{3}{8}$</p> 	<p>Complete method of rearranging to make z the subject.</p> $z = \frac{iw}{(1-w)}$ <p>Putting z in terms of their $w = 3$</p> <p>Applies $w = u + iv$, and uses Pythagoras correctly to get an equation in terms of u and v without any i's.</p> <p>Correct equation.</p> <p>Simplifies down to $u^2 + v^2 \pm \alpha u \pm \beta v \pm \delta = 0$.</p> <p>One of centre or radius correct. Both centre and radius correct.</p> <p>Circle indicated on the Argand diagram in the correct position in follow through quadrants. Ignore plotted coordinates.</p> <p>Region outside a circle indicated only.</p> <p>M1</p> <p>A1 aef</p> <p>dM1</p> <p>ddM1</p> <p>A1</p> <p>dddM1</p> <p>A1</p> <p>A1</p> <p>B1ft</p> <p>B1</p> <p>(8)</p> <p>(2)</p> <p>[10]</p>

Question Number	Scheme	Marks
Q7	$y = x^2 - a^2 , a > 1$ 	<p>Correct Shape. Ignore cusps. Correct coordinates.</p> <p>B1 B1</p>
(a)	$ x^2 - a^2 = a^2 - x, a > 1$ $\{ x > a\}, \quad x^2 - a^2 = a^2 - x$ $\Rightarrow x^2 + x - 2a^2 = 0$ $\Rightarrow x = \frac{-1 \pm \sqrt{1 - 4(1)(-2a^2)}}{2}$ $\Rightarrow x = \frac{-1 \pm \sqrt{1 + 8a^2}}{2}$ $\{ x < a\}, \quad -x^2 + a^2 = a^2 - x$ $\{\Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0\}$ $\Rightarrow x = 0, 1$	<p>(2)</p> <p>M1 aef</p> <p>Applies the quadratic formula or completes the square in order to find the roots.</p> <p>Both correct “simplified down” solutions.</p> <p>M1 aef</p> <p>M1 aef</p> <p>$x = 0$ $x = 1$</p> <p>B1 A1</p> <p>(6)</p>
(c)	$ x^2 - a^2 > a^2 - x, a > 1$ $x < \frac{-1 - \sqrt{1 + 8a^2}}{2} \quad \{\text{or}\} \quad x > \frac{-1 + \sqrt{1 + 8a^2}}{2}$ $\{\text{or}\} \quad 0 < x < 1$	<p>x is less than their least value x is greater than their maximum value</p> <p>B1 ft B1 ft</p> <p>For $\{ x < a\}$, Lowest $< x <$ Highest $0 < x < 1$</p> <p>M1 A1</p> <p>(4)</p> <p>[12]</p>

Question Number	Scheme	Marks
Q8	<p>$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$, $x = 0$, $\frac{dx}{dt} = 2$ at $t = 0$.</p> <p>(a) AE, $m^2 + 5m + 6 = 0 \Rightarrow (m + 3)(m + 2) = 0$ $\Rightarrow m = -3, -2$.</p> <p>So, $x_{CF} = Ae^{-3t} + Be^{-2t}$</p> <p>$\left\{ x = ke^{-t} \Rightarrow \frac{dx}{dt} = -ke^{-t} \Rightarrow \frac{d^2x}{dt^2} = ke^{-t} \right\}$</p> <p>$\Rightarrow ke^{-t} + 5(-ke^{-t}) + 6ke^{-t} = 2e^{-t} \Rightarrow 2ke^{-t} = 2e^{-t}$ $\Rightarrow k = 1$</p> <p>$\{ \text{So, } x_{PI} = e^{-t} \}$</p> <p>So, $x = Ae^{-3t} + Be^{-2t} + e^{-t}$</p> <p>$\frac{dx}{dt} = -3Ae^{-3t} - 2Be^{-2t} - e^{-t}$</p> <p>$t = 0, x = 0 \Rightarrow 0 = A + B + 1$ $t = 0, \frac{dx}{dt} = 2 \Rightarrow 2 = -3A - 2B - 1$</p> <p>$\left\{ \begin{array}{l} 2A + 2B = -2 \\ -3A - 2B = 3 \end{array} \right\}$</p> <p>$\Rightarrow A = -1, B = 0$</p> <p>So, $x = -e^{-3t} + e^{-t}$</p>	<p>$Ae^{m_1t} + Be^{m_2t}$, where $m_1 \neq m_2$. $Ae^{-3t} + Be^{-2t}$</p> <p>M1 A1</p> <p>Substitutes ke^{-t} into the differential equation given in the question. Finds $k = 1$.</p> <p>M1 A1</p> <p>their x_{CF} + their x_{PI}</p> <p>M1*</p> <p>Finds $\frac{dx}{dt}$ by differentiating their x_{CF} and their x_{PI}</p> <p>dM1*</p> <p>Applies $t = 0, x = 0$ to x and $t = 0, \frac{dx}{dt} = 2$ to $\frac{dx}{dt}$ to form simultaneous equations.</p> <p>ddM1*</p> <p>$x = -e^{-3t} + e^{-t}$</p> <p>A1 cao (8)</p>

Question Number	Scheme	Marks
(b)	$x = -e^{-3t} + e^{-t}$ $\frac{dx}{dt} = 3e^{-3t} - e^{-t} = 0$ $3 - e^{2t} = 0$ $\Rightarrow t = \frac{1}{2} \ln 3$ <p>So, $x = -e^{-\frac{3}{2} \ln 3} + e^{-\frac{1}{2} \ln 3} = -e^{\ln 3^{-\frac{3}{2}}} + e^{\ln 3^{-\frac{1}{2}}}$</p> $x = -3^{-\frac{3}{2}} + 3^{-\frac{1}{2}}$ $= -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$ $\frac{d^2x}{dt^2} = -9e^{-3t} + e^{-t}$ <p>At $t = \frac{1}{2} \ln 3$, $\frac{d^2x}{dt^2} = -9e^{-\frac{3}{2} \ln 3} + e^{-\frac{1}{2} \ln 3}$</p> $= -9(3)^{-\frac{3}{2}} + 3^{-\frac{1}{2}} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}}$ <p>As $\frac{d^2x}{dt^2} = -\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} = \left\{ -\frac{2}{\sqrt{3}} \right\} < 0$ then x is maximum.</p>	<p>Differentiates their x to give $\frac{dx}{dt}$ and puts $\frac{dx}{dt}$ equal to 0. M1</p> <p>A credible attempt to solve. $t = \frac{1}{2} \ln 3$ or $t = \ln \sqrt{3}$ or awrt 0.55 dM1* A1</p> <p>Substitutes their t back into x and an attempt to eliminate out the \ln's. ddM1</p> <p>uses exact values to give $\frac{2\sqrt{3}}{9}$ A1 AG</p> <p>Finds $\frac{d^2x}{dt^2}$ and substitutes their t into $\frac{d^2x}{dt^2}$ dM1*</p> <p>$-\frac{9}{3\sqrt{3}} + \frac{1}{\sqrt{3}} < 0$ and maximum conclusion. A1</p> <p>(7)</p> <p>[15]</p>