Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel International Advanced Level

Centre Number | Candidate Number

Monday 14 January 2019

Afternoon (Time: 1 hour 30 minutes) | Paper Reference **WFM01/01**

Further Pure Mathematics F1

Advanced/Advanced Subsidiary

You must have:
Mathematical Formulae and Statistical Tables (Blue)

Total Marks

Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B). Coloured pencils and highlighter pens must not be used.
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- When a calculator is used, the answer should be given to an appropriate degree of accuracy.

Information

- The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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1. The point A (12, 12) lies on the parabola with equation $y^2 = 12x$. The point S is the focus of this parabola. The line l passes through A and S.

(a) Find an equation of the line l.

(3)

The line l meets the directrix of the parabola at the point B.

(b) Find the coordinates of B.

(3)

January 2019
WFM01 Further Pure Mathematics F1
Mark Scheme

Question Number	Scheme	Notes	Marks
1.	$A(12, 12)$ lies on $y^2 = 12x$. l passes through A and S l meets the directrix of the parabola at B		
(a)	$\{a = 3 \Rightarrow S$ has coordinates$\}$ $(3, 0)$	**Either** states or uses $(3, 0)$ Can be implied by later work	B1
	<u>Way 1</u> **Both** $m_l = \dfrac{12}{12 - "3"}$ **and either** • $y = \dfrac{12}{12-"3"}(x - "3")$ **or** • $0 = \dfrac{12}{12-"3"}("3") + c \Rightarrow y = \dfrac{12}{12-"3"}x$ + their c **or** • $12 = \dfrac{12}{12-"3"}(12) + c \Rightarrow y = \dfrac{12}{12-"3"}x$ + their c	**<u>Way 1</u>** Correct method for finding the gradient between their S and $(12, 12)$ **and** a correct method for finding the equation of l	M1
	<u>Way 2</u> $\begin{cases} 3m + c = 0 \\ 12m + c = 12 \end{cases} \Rightarrow m = ..., c = ...$ and $y = $(their m)x + their c	**<u>Way 2</u>** Uses $y = mx + c$, their S and $(12, 12)$ to write two linear equations. Finds $m = ..., c = ...$ and writes $y = $(their m)x + their c	
	e.g. l: $y = \dfrac{12}{9}(x-3)$, $y = \dfrac{4}{3}x - 4$, $y - 12 = \dfrac{12}{9}(x - 12)$, $4x - 3y - 12 = 0$ **or** $3y = 4x - 12$	Any correct form for the equation of l which can be simplified or un-simplified **Note: ignore subsequent working** following on from a correct answer seen	A1
	Note: At least one of either x_S or y_S must be correct in order to gain M1		**(3)**
(b)	{directrix has equation} $x = -3$	**Either** states or uses $x = -3$ **or** states or uses $x = -$(their a), $a > 0$ where a is the x-coordinate of their S	M1
	$y = \dfrac{12}{9}(-3-3)$ $\{= -8\}$	**dependent on the previous M1 mark** Substitutes $x = -3$ into their equation of l **or** substitutes $x = -a$, $a > 0$ where a is the x-coordinate of their S into their equation of l. **Note:** l must represent a line (and not a curve) for this mark to be awarded **Note:** This mark may be implied by their y-coordinate	dM1
	{coordinates of B are} $(-3, -8)$	$(-3, -8)$	A1
			(3)
			6

		Question 1 Notes
1. (a)	**Note**	Give B0 for $a = 3$ by itself without reference to $(3, 0)$
	Note	Give B1 in part (a) for $S(3, 0)$ (and not $(3, 0)$) stated in part (b)
(b)	**Note**	Give 1st M1 for stating the x-coordinate of B as -3 or the x-coordinate of B as $-(\text{their } a)$, $a > 0$ where a is the x-coordinate of their S E.g. Give 1st M1 for $B(-3, ...)$
	Note	Give A0 for $x = -3,\ y = -8$ without reference to $(-3, -8)$
	Note	Give A0 for $x = -3,\ y = -8$ followed by $(-8, -3)$
	Note	Give A0 if more than one set of coordinates are given for B
(a), (b)	**Note**	Give B1 for a sketch with either 3 or $(3, 0)$ marked on the x-axis
	Note	Give 1st M1 in part (b) for a sketch with a vertical line drawn at $x = -3$ with -3 indicated
	Note	Give 1st M1 in part (b) a statement "directrix is $x = -3$" seen anywhere

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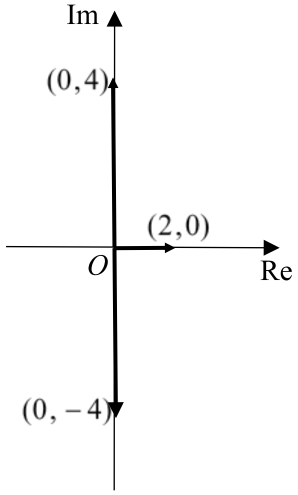
2.

$$f(z) = z^3 - 2z^2 + 16z - 32$$

(a) Show that $f(2) = 0$ **(1)**

(b) Use algebra to solve $f(z) = 0$ completely. **(3)**

(c) Show, on a single Argand diagram, all three roots of the equation $f(z) = 0$ **(2)**

Question Number	Scheme	Notes	Marks
2.	$\text{f}(z) = z^3 - 2z^2 + 16z - 32$		
(a)	• $\{\text{f}(2) = \}\ 8 - 8 + 32 - 32 = 0$ **or** • $\{\text{f}(2) = \}\ (2)^3 - 2(2)^2 + 16(2) - 32 = 0$	Uses working to show that $\text{f}(2) = 0$	B1
			(1)
(b)	$\{\text{f}(z) = \}\ (z-2)(z^2+16)$	Uses only $(z-2)$ to find a quadratic factor. e.g. using long division with $(z-2)$ to get as far as $z^2 + ...$ or factorising to give $(z-2)(z^2 + ...)$ **Note:** 1st M1 can be given for sight of a correct (z^2+16)	M1
	$\{(z^2+16) = 0 \Rightarrow z = \}\ \pm 4\text{i}$	Correct method of solving their quadratic factor	M1
	$\{\text{f}(z) = 0 \Rightarrow z = \}\ 2,\ 4\text{i},\ -4\text{i}$	$2,\ 4\text{i}$ **and** -4i	A1
			(3)
(c)		<u>**Criteria**</u> • The number 2 plotted correctly on the positive real axis • **dependent on a correct method for solving their quadratic factor or dependent on deducing correct roots of 2, 4i, - 4i** Their final two roots of the form $\pm\mu\text{i},\ \mu \neq 0$ or the form $\lambda \pm \mu\text{i},\ \mu \neq 0,$ are plotted correctly	
		Satisfies at least one of the criteria	B1ft
		Only 3 roots plotted, satisfying both criteria with some indication of scale or coordinates stated. **Note:** The pair of complex roots should be approximately symmetrical about the real axis **Note:** Condone the labels $4\text{i},\ -4\text{i}$ marked on the y-axis	B1ft
			(2)
			6

	Question 2 Notes	
2. (b)	**Note**	You can assume $x \equiv z$ for solutions in this part
	Note	No algebraic working leading to $z = 2,\ 4\text{i},\ -4\text{i}$ is M0 M0 A0
	Note	Allow M1 M1 A1 for $(z-2)(z+4\text{i})(z-4\text{i})\ \{=0\} \Rightarrow z = 2,\ 4\text{i},\ -4\text{i}$
	Note	Allow M1 M0 A0 for $(z-2)(z+4\text{i})(z-4\text{i})\ \{=0\}$ by itself, but please note that you cannot recover the final M1 A1 marks for work seen in part (c)
	Note	Give M1 M0 A0 for $(z-2)(z^2+16)\ \{=0\} \Rightarrow (z-2)(z+4\text{i})(z-4\text{i})\ \{=0\}$ by itself, but please note that you cannot recover the final M1 A1 marks for work seen in part (c)
	Note	$z = \pm\sqrt{16\text{i}}$ unless recovered is 2nd M0 1st A0
	Note	Give 2nd M1 for $z^2 + k = 0,\ k > 0 \Rightarrow$ **at least one of either** $z = \sqrt{k}\,\text{i}$ or $z = -\sqrt{k}\,\text{i}$ So, e.g. give 2nd M1 for $z^2 + 16 = 0 \Rightarrow z = 4\text{i}$
	Note	Give 2nd M0 for $z^2 + k = 0,\ k > 0 \Rightarrow z = \pm k\text{i}$
	Note	Give 2nd M0 for $z^2 + k = 0,\ k > 0 \Rightarrow z = \pm k$ or $z = \pm\sqrt{k}$
	Note	Give 2nd M1 for $z^2 - k = 0,\ k > 0 \Rightarrow$ **both** $z = \sqrt{k}$ **and** $z = -\sqrt{k}$
	Note	<u>**Special Case:**</u> If ***their quadratic*** factor $z^2 + "a"z + "b"$ **can** be factorised then give Special Case 2nd M1 for correct factorisation leading to $\boldsymbol{z} = ...$ Otherwise, give 2nd M0 for applying a method of factorisation to solve their 3TQ.

		Question 2 Notes Continued
2. (b)	**Note**	**Reminder:** Method Mark for solving a 3TQ, "$az^2 + bz + c = 0$" **Formula:** Attempt to use the correct formula (with values for a, b and c) **Completing the square:** $\left(z \pm \frac{b}{2}\right)^2 \pm q \pm c = 0,\ q \neq 0$, leading to $z = ...$
	Note	Send to review solutions involving α, β, γ roots. E.g. $-2 = -(\alpha + \beta + \gamma)$
(c)	**Note**	Drawing the lines $z = 2$, $z = 4\text{i}$, $z = -4\text{i}$ instead of plotting the points $(2, 0)$, $(0, 4)$ and $(0, -4)$ is B0 B0
	Note	Indication of coordinates includes stating e.g. $z_1 = 2$, $z_2 = 4\text{i}$, $z_3 = -4\text{i}$ and plotting z_1, z_2 and z_3 in their relevant positions on an Argand diagram
(b), (c)	**Note**	You cannot recover work for part (b) in part (c)

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3. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} (2r+5)^2 = \frac{n}{3}\left[(an+b)^2 + c\right]$$

where a, b and c are integers to be found.

(5)

(b) Use the answer to part (a) to evaluate $\sum_{r=0}^{100} (2r+5)^2$

(2)

Question Number	Scheme	Notes	Marks
3. (a)	$\sum_{r=1}^{n}(2r+5)^2 = 4\sum_{r=1}^{n}r^2 + 20\sum_{r=1}^{n}r + \sum_{r=1}^{n}25$		
	$= 4\left(\frac{1}{6}n(n+1)(2n+1)\right) + 20\left(\frac{1}{2}n(n+1)\right) + 25n$	Attempts to expand $(2r+5)^2$ and attempts to substitute at least one formula for either $\sum_{r=1}^{n}r^2$ or $\sum_{r=1}^{n}r$ into their resulting expression	M1 (B1 on ePEN)
		$4\left(\frac{1}{6}n(n+1)(2n+1)\right) + 20\left(\frac{1}{2}n(n+1)\right)$ which can be simplified or un-simplified	A1 (M1 on ePEN)
		Use of $\sum_{r=1}^{n}1 = n$	B1
	$= \frac{1}{3}n\big(2(n+1)(2n+1)+30(n+1)+75\big)$	Obtains an expression of the form $\alpha n(n+1)(2n+1)+\beta n(n+1)+\lambda n;\ \alpha,\beta,\lambda\neq 0$ and attempts to factorise out at least n	M1
	$= \frac{1}{3}n(4n^2+6n+2+30n+30+75)$		
	$= \frac{n}{3}(4n^2+36n+107)$		
	$= \frac{n}{3}\left[(2n+9)^2+26\right]$ $\left\{\text{or } \frac{n}{3}\left[(-2n-9)^2+26\right]\right\}$	Correct completion **Note:** $a=2, b=9$ and $c=26$ or $a=-2, b=-9$ and $c=26$	A1
			(5)
(b)	$\left\{\sum_{r=0}^{100}(2r+5)^2 = \right\}$ $= \frac{100}{3}\left[(2(100)+9)^2+26\right]+(5)^2$	Substitutes $n=100$ into their expression for $\sum_{r=1}^{n}(2r+5)^2$ which is in terms of n, and adds $(5)^2$ or 25 or $(2(0)+5)^2$ o.e. to the result	M1
	$\left\{= \frac{100}{3}(43707)+25\right\} = 1456925$	Obtains 1456925	A1
			(2)
			7

Question 3 Notes

3. (a)	**Note**	Applying e.g. $n=1$, $n=2$ and $n=3$ to the printed equation without applying the standard formulae to give $a=2, b=9$ and $c=26$ is M0 A0 B0 M0 A0
	Alt 1	**Alt Method 1 (Award the first three marks using the main scheme)** Using $\frac{4}{3}n^3+12n^2+\frac{107}{3}n \equiv \frac{a^2}{3}n^3+\frac{2ab}{3}n^2+\frac{b^2+c}{3}n$ o.e.
	M1	Equating coefficients to find at least two of $a=\ldots, b=\ldots$ or $c=\ldots$ and at least one of **either** $a=2, b=9$ or $c=26$ **or** $a=-2, b=-9$ and $c=26$
	A1	Finds $a=2, b=9$ and $c=26$ or $a=-2, b=-9$ and $c=26$
	Note	Allow final M1A1 for $\frac{4}{3}n^3+12n^2+\frac{107}{3}n \rightarrow \frac{n}{3}\left[(2n+9)^2+26\right]$ with no incorrect working.
	Note	A correct proof of $\sum_{r=1}^{n}(2r+5)^2 = \frac{n}{3}\left[(2n+9)^2+26\right]$ followed by stating an incorrect e.g. $a=9, b=2$ and $c=26$ is M1 A1 B1 M1 A1 (ignore subsequent working)

	Question 3 Notes Continued	
3. (b)	**Note**	Allow M1 for $\frac{100}{3}(4(100)^2+36(100)+107)+(5)^2$ and A1 for obtaining 1456925
	Note	Allow M1 for $4\left(\frac{1}{6}(100)(101)(201)\right)+20\left(\frac{1}{2}(100)(101)\right)+25(100)+(5)^2$ $\{=1353400+101000+2500+25\}$ and A1 for obtaining 1456925
	Note	**dependent on obtaining 1st M1, 1st A1 and B1 in part (a)** Allow M1 A1 for $1456900+25=1456925$
	Note	Give M0 A0 for writing down 1456925 by itself with no supporting working
	Note	Give M0 A0 for listing individual terms i.e $\sum_{r=0}^{100}(2r+5)^2=5^2+7^2+9^2+11^2+\ldots+205^2=1456925$, by itself is M0 A0
	Note	Give M0 A0 for applying $\frac{100}{3}\left[(2(100)+9)^2+26\right]+\frac{(-1)}{3}\left[(2((-1))+9)^2+26\right]=1456900--25=1456925$

Leave blank

4.

$$f(x) = 2x^3 - \frac{7}{x^2} + 16, \qquad x \neq 0$$

The equation $f(x) = 0$ has a single root α between $x = -2$ and $x = -1$

(a) Starting with the interval $[-2, -1]$, use interval bisection twice to find an interval of width 0.25 that contains α.

(3)

The equation $f(x) = 0$ also has a single root β in the interval $[0.6, 0.7]$.

(b) Taking 0.65 as a first approximation to β, apply the Newton-Raphson procedure once to $f(x)$ to obtain a second approximation to β, giving your answer to 4 decimal places.

(4)

Question Number	Scheme	Notes	Marks
4.	Given $\text{f}(x) = 2x^3 - \frac{7}{x^2} + 16$, $x \neq 0$; Roots α, β: $-2 \leq \alpha \leq -1$ and $0.6 \leq \beta \leq 0.7$		
(a) **Way 1**	$\text{f}(-1.5) = ...$	Attempts to evaluate $\text{f}(-1.5)$	M1
	$\text{f}(-1.75) = ...$	**dependent on the previous M mark** Evaluates $\text{f}(-1.75)$ (and not $\text{f}(-1.25)$)	dM1
	$\text{f}(-2) = -1.75$ **or** $\text{f}(-1) = 7$ $\text{f}(-1.5) = 6.1388...$ or $\frac{221}{36}$ $\text{f}(-1.75) = 2.9955...$ or $\frac{671}{224}$ so interval is $[-2, -1.75]$	**dependent on the 2 previous marks** **Both** • $\text{f}(-2)$ correct or correct awrt (or truncated) to 1sf or $\text{f}(-1) = 7$ **and** • $\text{f}(-1.5)$ and $\text{f}(-1.75)$ correct or correct awrt (or truncated) to 1 sf **and** the correct interval stated **Note:** Allow $-2 \leq x \leq -1.75$ or $-2 < x < -1.75$ or $-2 \leq \alpha \leq -1.75$ or $-2 < \alpha < -1.75$ or $[-2, -1.75]$ or $(-2, -1.75)$ equivalent in words. Condone $-2 - -1.75$ Allow a mixture of "ends". Do not allow incorrect statements such as $-1.75 < \alpha < -2$ or $(-1.75, -2)$ or $-1.75 - -2$ unless they are recovered. Ignore the subsequent iteration of $\text{f}(-1.875)$	A1
	Note that some candidates only indicate the sign of f and not its value. In this case the M marks can still score as defined but not the A mark.		**(3)**
(a) **Way 2**	**Common approach in the form of a table (use the mark scheme above)**		

a	$\text{f}(a)$	b	$\text{f}(b)$	$\frac{a+b}{2}$	$\text{f}\left(\frac{a+b}{2}\right)$
-2	-1.75	-1	7	-1.5	$6.1388...$
-2	-1.75	-1.5	$6.1388...$	-1.75	$2.9955...$

so interval is $-2 \leq \alpha \leq -1.75$ **would score full marks in part (a)**

Question Number	Scheme	Notes	Marks
(b)	$\text{f}'(x) = 6x^2 + 14x^{-3}$	At least one of either $2x^3 \to \pm Ax^2$ or $-\frac{7}{x^2} \to \pm Bx^{-3}$; $A, B \neq 0$	M1
		Correct differentiation which can be simplified or un-simplified	A1
	$\left\{\beta \simeq 0.65 - \frac{\text{f}(0.65)}{\text{f}'(0.65)}\right\} \Rightarrow \beta \simeq 0.65 - \frac{-0.01879733728...}{53.51360719...}$	**dependent on the previous M mark** Valid attempt at Newton-Raphson using their values of $\text{f}(0.65)$ and $\text{f}'(0.65)$	dM1
	$\{\beta = 0.6503512623...\} \Rightarrow \beta = 0.6504$ (4 dp)	**dependent on all 3 previous marks** 0.6504 on their first iteration (Ignore any subsequent iterations)	A1 **cso cao**
	Correct differentiation followed by a correct answer of 0.6504 scores full marks in part (b) **Correct answer with <u>no</u> working scores no marks in part (b)**		**(4)**
			7

	Question 4 Notes	
4. (a)	**Note**	Give 2nd M0 and A0 for evaluating both $\text{f}(-1.25)$ and $\text{f}(-1.75)$
	Note	Do not allow "interval $=\text{f}(-2)$ to $\text{f}(-1.75)$" unless recovered.
	Note	A method of evaluating $\text{f}(-1.5)$ followed by $\text{f}(-1.75)$ with ***no evidence*** of evaluating at least one of either $\text{f}(-2)$ or $\text{f}(-1)$ is M1 dM1 A0.
	Note	Do not confuse the -1.75 in $\text{f}(-2) = -1.75$ with the -1.75 in $(-2, -1.75)$

	Question 4 Notes Continued	
4. (b)	**dM1**	This mark can be implied by applying at least one correct ***value*** of either $\mathrm{f}(0.65)$ or their $\mathrm{f}'(0.65)$ (where $\mathrm{f}'(0.65)$ is found using their $\mathrm{f}'(x)$) to 1 significant figure in $0.65 - \dfrac{\mathrm{f}(0.65)}{\mathrm{f}'(0.65)}$. So just $0.65 - \dfrac{\mathrm{f}(0.65)}{\mathrm{f}'(0.65)}$ with an incorrect answer and no other evidence scores final dM0A0.
	Note	If you see $0.65 - \dfrac{\mathrm{f}(0.65)}{\mathrm{f}'(0.65)} = 0.6504$ with no algebraic differentiation, then send the response to review.
	Note	You can imply the M1 A1 marks for algebraic differentiation for either • $\mathrm{f}'(0.65) = 6(0.65)^2 + 14(0.65)^{-3}$ • $\mathrm{f}'(0.65)$ applied correctly in $\beta \simeq 0.65 - \dfrac{2(0.65)^3 - \dfrac{7}{(0.65)^2} + 16}{6(0.65)^2 + 14(0.65)^{-3}}$
	Note	**Differentiating INCORRECTLY to give** $\mathrm{f}'(x) = 6x^2 - 14x^{-3}$ leads to $\beta \simeq 0.65 - \dfrac{-0.01879733728...}{-48.44360719...} = 0.6496119749... = 0.6496$ (4 dp) **This response should be awarded M1 A0 dM1 A0**
	Note	**Differentiating INCORRECTLY to give** $6x^2 - 14x^{-3}$ and $\beta \simeq 0.65 - \dfrac{\mathrm{f}(0.65)}{\mathrm{f}'(0.65)} = 0.6496$ is **M1 A0 dM1 A0**

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5. The rectangular hyperbola H has equation $xy = 16$

The point P, on H, has coordinates $\left(4p, \frac{4}{p}\right)$ where p is a non-zero constant.

(a) Show, using calculus, that the tangent to H at the point P has equation

$$x + p^2y = 8p$$

(4)

Given that the tangent to H at the point P passes through the point (7, 1)

(b) use algebra to find the coordinates of the two possible positions of P.

(4)

Question Number	Scheme	Notes	Marks
5.	$H: xy = 16;\ P\left(4p, \frac{4}{p}\right),\ p \neq 0,$ lies on H. Tangent to H at P passes through the point $(7, 1)$		
(a)	$y = \frac{16}{x} = 16x^{-1} \Rightarrow \frac{dy}{dx} = -16x^{-2}$ or $-\frac{16}{x^2}$	$\frac{dy}{dx} = \pm k x^{-2};\ k \neq 0$	
	$xy = 16 \Rightarrow x\frac{dy}{dx} + y = 0$	Uses implicit differentiation to give $\pm x\frac{dy}{dx} \pm y$	M1
	$x = 4t,\ y = \frac{4}{t} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = -\left(\frac{4}{t^2}\right)\left(\frac{1}{4}\right)$	their $\frac{dy}{dt} \times \frac{1}{\text{their } \frac{dy}{dt}}$; **Condone** $t \equiv p$	
	So at P, $m_T = -\frac{1}{p^2}$	Correct calculus work leading to $m_T = -\frac{1}{p^2}$	A1
	• $y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$ **or** • $\frac{4}{p} = -\frac{1}{p^2}(4p) + c \Rightarrow y = -\frac{1}{p^2}x +$ their c	Correct straight line method for an equation of the tangent where $m_T\left(\neq \frac{-1}{\text{their } m_T} \text{ or } \neq \frac{1}{\text{their } m_T}\right)$ is found by using calculus. **Note:** m_T must be a function of p **Note:** Condone (slip) of using $m_T = -(\text{their } m_T)$	M1
	Correct algebra leading to $x + p^2 y = 8p$ *	Correct solution only	A1 *
			(4)
(b)	$\{(7, 1) \Rightarrow\}\ \ 7 + p^2 = 8p$	Substitutes $x = 7,\ y = 1$ into the given equation or their answer to part (a). **Note:** Condone substituting $x = 1,\ y = 7$ into the given equation or their answer to part (a) for M1	M1
	$\{\Rightarrow p^2 - 8p + 7 = 0\}$		
	$(p - 7)(p - 1) = 0 \Rightarrow p = ...$	**dependent on the previous M mark** Correct method (e.g. factorising, applying the quadratic formula or completing the square) of solving a 3TQ to find $p = ...$	dM1
	$\{p = 1 \Rightarrow\}\ x = 4,\ y = 4$ $\{p = 7 \Rightarrow\}\ x = 28,\ y = \frac{4}{7}$ or awrt 0.57	**dependent on substituting $x = 7, y = 1$ into the given equation or their answer to part (a)** Obtains at least one correct set of corresponding values for $x = ...$ **and** $y = ...$	A1
	$\{\text{So } P \text{ can be}\}\ \ (4, 4), \left(28, \frac{4}{7}\right)$	Both correct sets of coordinates of B	A1
			(4)
			8

		Question 5 Notes
5. (a)	**Note**	Allow $yp^2 + x = 8p$ or $8p = x + p^2y$ or $8p = p^2y + x$ for the final A1
(b)	**Note**	Do not confuse $(7, 1)$ or $x = 7,\ y = 1$ with $p = 7, 1$
	Note	A decimal answer of e.g. $(4, 4), (28, 0.57)$ (without a correct exact answer) is 2nd A0
	Note	Imply the dM1 mark for ***writing down the correct*** roots for ***their*** quadratic equation. . E.g. $7 + p^2 = 8p$ or $p^2 - 8p + 7 = 0 \rightarrow p = 7, 1$
	Note	E.g. give dM0 for $7 + p^2 = 8p$ or $p^2 - 8p + 7 = 0 \rightarrow p = -7, -1$ [incorrect solution] with NO INTERMEDIATE working.
	Note	Give M1 dM1 A1 for either • $7 + p^2 = 8p \rightarrow x = 4,\ y = 4$ or $(4, 4)$ • $7 + p^2 = 8p \rightarrow x = 28,\ y = \frac{4}{7}$ or awrt 0.57 **or** $\left(28, \frac{4}{7}\right)$ **or** $(28, \text{awrt } 0.57)$ with NO INTERMEDIATE working.
	Note	Give M1 dM1 A1 A1 for • $7 + p^2 = 8p \rightarrow (4, 4), \left(28, \frac{4}{7}\right)$ with NO INTERMEDIATE working.
	Note	Give M0 dM0 A0 A0 for writing down $(4, 4), \left(28, \frac{4}{7}\right)$ with no prior working.
	Note	Only a maximum of M1 dM1 A0 A0 can be scored for substituting for $x = 1,\ y = 7$ (and not $x = 7,\ y = 1$) into $x + p^2y = 8p$ **Note:** $x = 1,\ y = 7 \Rightarrow 1 + 7p^2 = 8p \Rightarrow (7p - 1)(p - 1) \Rightarrow p = \frac{1}{7}, 1 \Rightarrow \left(\frac{4}{7}, 28\right), (4, 4)$
	Note	**<u>Alt 1 Method</u>** • $x = 7,\ y = 1 \Rightarrow 7 + p^2 = 8p \Rightarrow (p - 1)(p - 7) \Rightarrow p = 1, 7$ • $p = 1 \Rightarrow x + (1)y = 8(1)$ and $x + \frac{16}{x} = 8 \Rightarrow x^2 - 8x + 16 = 0 \Rightarrow (x - 4)(x - 4) = 0$ $\Rightarrow x = 4,\ y = 4 \Rightarrow (4, 4)$ • $p = 7 \Rightarrow x + 49y = 56$ and $x + 49\left(\frac{16}{x}\right) = 56 \Rightarrow x^2 - 56x + 784 = 0 \Rightarrow (x - 28)(x - 28) = 0$ $\Rightarrow x = 28,\ y = \frac{4}{7} \Rightarrow \left(28, \frac{4}{7}\right)$
	Note	**<u>Incorrect method of substituting $xy = 16$ and (7, 1) into $x + p^2y = 8p$</u>** Give M0 dM0 A0 A0 for • $x + p^2\left(\frac{16}{x}\right) = 8p$ and $x = 7 \Rightarrow 7 + \frac{16}{7}p^2 = 8p \Rightarrow 16p^2 - 56p + 49 = 0 \Rightarrow (4p - 7)(4p - 7) = 0$ $\Rightarrow p = \frac{7}{4} \Rightarrow x = 7,\ y = \frac{16}{7} \Rightarrow \left(7, \frac{16}{7}\right)$ • $\frac{16}{y} + p^2y = 8p$ and $y = 1 \Rightarrow 16 + p^2 = 8p \Rightarrow p^2 - 8p + 16 = 0 \Rightarrow (p - 4)(p - 4) = 0$ $\Rightarrow p = 4 \Rightarrow x = 16,\ y = 1 \Rightarrow (16, 1)$
	Note	Give M1 dM0 A0 A0 for • $x = 7,\ y = 1$ into $x + p^2y = 8p \Rightarrow 7 + p^2 = 8 \Rightarrow (p + 1)(p - 1) \Rightarrow p = 1, -1 \Rightarrow (4, 4), (-4, -4)$

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6. It is given that α and β are roots of the equation

$$12x^2 - 3x + 4 = 0$$

Without solving the equation,

(a) find the exact value of $\frac{2}{\alpha} + \frac{2}{\beta}$

(3)

(b) find a quadratic equation that has roots $\frac{2}{\alpha} - \beta$ and $\frac{2}{\beta} - \alpha$ giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers to be found.

(6)

Question Number	Scheme	Notes	Marks
6.	$12x^2 - 3x + 4 = 0$ has roots α, β		
(a)	$\alpha + \beta = \frac{3}{12}$ or $\frac{1}{4}$, $\alpha\beta = \frac{4}{12}$ or $\frac{1}{3}$	**Both** $\alpha + \beta = \frac{3}{12}$ or $\frac{1}{4}$ **and** $\alpha\beta = \frac{4}{12}$ or $\frac{1}{3}$, seen or implied	B1
	$\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2\beta + 2\alpha}{\alpha\beta}$	**States or uses** $\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2\beta + 2\alpha}{\alpha\beta}$ or $\frac{2(\alpha+\beta)}{\alpha\beta}$	M1
	$= \frac{2\left(\frac{3}{12}\right)}{\left(\frac{4}{12}\right)} = \frac{3}{2}$	**dependent on BOTH previous marks being awarded** $\frac{3}{2}$ or $\frac{6}{4}$ or 1.5 **from correct working**	A1 **cso cao**
			(3)
(b)	Sum $= \frac{2}{\alpha} - \beta + \frac{2}{\beta} - \alpha$ $= \frac{2}{\alpha} + \frac{2}{\beta} - (\alpha + \beta)$ $= \frac{3}{2} - \frac{1}{4} = \frac{5}{4}$	Uses at least one of their $\frac{2}{\alpha} + \frac{2}{\beta}$ or their $(\alpha + \beta)$ in an attempt to find a **numerical value** for the sum of $\left(\frac{2}{\alpha} - \beta\right)$ and $\left(\frac{2}{\beta} - \alpha\right)$	M1
		Correct sum of $\frac{5}{4}$ or $\frac{15}{12}$ or 1.25 which can be implied	A1
	Product $= \left(\frac{2}{\alpha} - \beta\right)\left(\frac{2}{\beta} - \alpha\right)$ $= \frac{4}{\alpha\beta} - 2 - 2 + \alpha\beta$ $= \frac{4}{\left(\frac{1}{3}\right)} - 2 - 2 + \frac{1}{3} = \frac{25}{3}$	Expands $\left(\frac{2}{\alpha} - \beta\right)\left(\frac{2}{\beta} - \alpha\right)$ to give $\frac{P}{\alpha\beta} + Q + R\alpha\beta$; $P, Q, R \neq 0$ and uses their $\alpha\beta$ at least once in an attempt to find a **numerical value** for the product of $\left(\frac{2}{\alpha} - \beta\right)$ and $\left(\frac{2}{\beta} - \alpha\right)$	M1
		Correct product of $\frac{25}{3}$ or $8\frac{1}{3}$ or $8.\dot{3}$ or $\frac{100}{12}$	A1
	$x^2 - \frac{5}{4}x + \frac{25}{3} = 0$	Applies $x^2 - (\text{sum})x + \text{product}$ (can be implied), where sum and product are numerical values. **Note:** " $=0$ " is not required for this mark	M1
	$12x^2 - 15x + 100 = 0$	***Any integer multiple*** of $12x^2 - 15x + 100 = 0$, including the " $=0$ "	A1 **cso**
			(6)
			9

Question 6 Notes

6. (a)	**Note**	Writing down $\alpha, \beta = \frac{3 + \sqrt{183}\,\mathrm{i}}{24}, \frac{3 - \sqrt{183}\,\mathrm{i}}{24}$ and then stating $\alpha + \beta = \frac{1}{4}$, $\alpha\beta = \frac{1}{3}$ **or** applying $\alpha + \beta = \frac{3 + \sqrt{183}\,\mathrm{i}}{24} + \frac{3 - \sqrt{183}\,\mathrm{i}}{24} = \frac{1}{4}$ and $\alpha\beta = \left(\frac{3 + \sqrt{183}\,\mathrm{i}}{24}\right)\left(\frac{3 - \sqrt{183}\,\mathrm{i}}{24}\right) = \frac{1}{3}$ scores B0
	Note	Those candidates who then apply $\alpha + \beta = \frac{4}{5}$, $\alpha\beta = \frac{3}{5}$, having written down/applied $\alpha, \beta = \frac{3 + \sqrt{183}\,\mathrm{i}}{24}, \frac{3 - \sqrt{183}\,\mathrm{i}}{24}$, can only score the M mark in part (a) for $\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2\beta + 2\alpha}{\alpha\beta}$
	Note	Give B0 M0 A0 for $\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2}{\left(\frac{3 + \sqrt{183}\,\mathrm{i}}{24}\right)} + \frac{2}{\left(\frac{3 - \sqrt{183}\,\mathrm{i}}{24}\right)} = \frac{3}{2}$

		Question 6 Notes Continued		
6. (a)	**Note**	Give B0 M1 A0 for $\frac{2}{\alpha}+\frac{2}{\beta}=\frac{2\beta+2\alpha}{\alpha\beta}=\frac{2\left(\frac{3-\sqrt{183}\,i}{24}\right)+2\left(\frac{3+\sqrt{183}\,i}{24}\right)}{\left(\frac{3+\sqrt{183}\,i}{24}\right)\left(\frac{3-\sqrt{183}\,i}{24}\right)}=\frac{3}{2}$		
	Note	Allow B1 for both $S=\frac{1}{4}$ **and** $P=\frac{1}{3}$ or for both $\sum=\frac{1}{4}$ **and** $\prod=\frac{1}{3}$		
(b)	**Note**	A correct method leading to $a=12,\ b=-15,\ c=100$ without writing a final answer of $12x^2-15x+100=0$ is final M1A0		
	Note	Using $\frac{3+\sqrt{183}\,i}{24},\ \frac{3-\sqrt{183}\,i}{24}$ ***explicitly*** to find the sum and product of $\left(\frac{2}{\alpha}-\beta\right)$ and $\left(\frac{2}{\beta}-\alpha\right)$ to give $x^2-\frac{5}{4}x+\frac{25}{3}=0\Rightarrow 12x^2-15x+100=0$ scores M0 A0 M0 A0 M1A0 in part (b)		
	Note	Using $\frac{3+\sqrt{183}\,i}{24},\ \frac{3-\sqrt{183}\,i}{24}$ to find $\alpha+\beta=\frac{1}{4},\ \alpha\beta=\frac{1}{3},\ \frac{2}{\alpha}+\frac{2}{\beta}=\frac{3}{2}$ **and applying** $\left\{\alpha+\beta=\frac{1}{4},\right\}\ \alpha\beta=\frac{1}{3},\ \frac{2}{\alpha}+\frac{2}{\beta}=\frac{3}{2}$ can potentially score full marks in part (b). E.g. Score M1 A1 M1 A1 M1 A1 for • Sum = = $\frac{2}{\alpha}-\beta+\frac{2}{\beta}-\alpha=\frac{2}{\alpha}+\frac{2}{\beta}-(\alpha+\beta)=\frac{3}{2}-\frac{1}{4}=\frac{5}{4}$ • Product = $\left(\frac{2}{\alpha}-\beta\right)\left(\frac{2}{\beta}-\alpha\right)=\frac{4}{\alpha\beta}-2-2+\alpha\beta=\frac{4}{\left(\frac{1}{3}\right)}-2-2+\frac{1}{3}=\frac{25}{3}$ • $x^2-\frac{5}{4}x+\frac{25}{3}=0\Rightarrow 12x^2-15x+100=0$		
	Note	<u>**Alternative method for finding the sum**</u> Sum = $\frac{2}{\alpha}-\beta+\frac{2}{\beta}-\alpha=\frac{2\beta-\alpha\beta^2+2\alpha-\alpha^2\beta}{\alpha\beta}=\frac{2(\alpha+\beta)-\alpha\beta(\beta+\alpha)}{\alpha\beta}$ $=\frac{2(\frac{1}{4})-(\frac{1}{3})(\frac{1}{4})}{(\frac{1}{3})}=\frac{\frac{1}{2}-\frac{1}{12}}{\frac{1}{3}}=\frac{\frac{5}{12}}{\frac{1}{3}}=\frac{15}{12}=\frac{5}{4}$		
	Note	<u>**Alternative method for finding the product**</u> Product = $\left(\frac{2}{\alpha}-\beta\right)\left(\frac{2}{\beta}-\alpha\right)$ $=\frac{(\alpha\beta-2)^2}{\alpha\beta}=\frac{((\frac{1}{3})-2)^2}{(\frac{1}{3})}$ $=\frac{\frac{25}{9}}{(\frac{1}{3})}=\frac{25}{3}$	Expands $\left(\frac{2}{\alpha}-\beta\right)\left(\frac{2}{\beta}-\alpha\right)$ to give $\frac{(\alpha\beta-2)^2}{\alpha\beta}$ and uses their $\alpha\beta$ at least once in an attempt to find a **numerical value** for the product of $\left(\frac{2}{\alpha}-\beta\right)$ and $\left(\frac{2}{\beta}-\alpha\right)$	M1
			Correct product of $\frac{25}{3}$ or $8\frac{1}{3}$ or $8.\dot{3}$	A1

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7.

$$\mathbf{P} = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

(a) Show that $\mathbf{P}^3 = 8\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.

(3)

(b) Describe fully the transformation represented by the matrix **P** as a combination of two simple geometrical transformations.

(4)

(c) Find the matrix $\mathbf{P}^{35}$, giving your answer in the form

$$\mathbf{P}^{35} = 2^k \begin{pmatrix} -1 & a \\ b & -1 \end{pmatrix}$$

where k is an integer and a and b are surds to be found.

(2)

Question Number	Scheme	Notes	Marks
7.	$\mathbf{P}=\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$; (a) $\mathbf{P}^3=8\mathbf{I}$; (c) $\mathbf{P}^{35}=2^k\begin{pmatrix}-1 & a\\ b & -1\end{pmatrix}$		
(a)	$\{\mathbf{P}^2=\}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}=\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$	Finds $\mathbf{P}^2$ (which can be un-simplified) with at least 3 correct elements for $\mathbf{P}^2$	M1
	$\{\mathbf{P}^3=\}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}=8\mathbf{I}$ * **or** $\{\mathbf{P}^3=\}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}=\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}=8\mathbf{I}$ *	**dependent on the previous M mark** Multiplies $\mathbf{P}^2$ by $\mathbf{P}$ or multiplies $\mathbf{P}$ by $\mathbf{P}^2$ to give a 2×2 matrix of 4 elements for $\mathbf{P}^3$ with at least 2 correct elements	dM1
		Correct proof with no errors	A1 *
			(3)
(b)	Enlargement	Enlargement or enlarge or dilation	M1
	Centre $(0, 0)$ with scale factor 2	about $(0, 0)$ or about O or about the origin **and** scale or factor or times **and** 2	A1
	Rotation	Rotation or rotate (condone turn)	M1
	120 degrees (anticlockwise) about $(0, 0)$	**Both** 120 degrees or $\frac{2\pi}{3}$ or 240 degrees clockwise or $\frac{4\pi}{3}$ clockwise **and** about $(0, 0)$ or about O or about the origin	A1
			(4)
(c) **Way 1**	$\mathbf{P}^{35}=(\mathbf{P}^3)^{11}\times\mathbf{P}^2$ **or** $\mathbf{P}^{35}=\mathbf{P}^{33}\times\mathbf{P}^2$		
	$=(8\mathbf{I})^{11}\times\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$ **or** $=(2\mathbf{I})^{33}\times\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$	$((8\mathbf{I})^{11}$ or $(8)^{11})\times$(their $\mathbf{P}^2$) or $((2\mathbf{I})^{33}$ or $(2)^{33})\times$(their $\mathbf{P}^2$)	M1
	$=2^{34}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$	Correct answer **Note:** $k=34$, $a=\sqrt{3}$, $b=-\sqrt{3}$	A1
			(2)
(c) **Way 2**	$\mathbf{P}^{35}=(\mathbf{P}^3)^{12}\times\mathbf{P}^{-1}$ **or** $\mathbf{P}^{35}=\mathbf{P}^{36}\times\mathbf{P}^{-1}$		
	$=(8\mathbf{I})^{12}\times\frac{1}{(-1)(-1)-(-\sqrt{3})(\sqrt{3})}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$ **or** $=(2\mathbf{I})^{36}\times\frac{1}{(-1)(-1)-(-\sqrt{3})(\sqrt{3})}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$	$((8\mathbf{I})^{12}$ or $(8)^{12})\times\frac{1}{\text{their det}(\mathbf{P})}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$ or $((2\mathbf{I})^{36}$ or $(2)^{36})\times\frac{1}{\text{their det}(\mathbf{P})}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$ where their $\det(\mathbf{P})>1$	M1
	$\left\{=(2^{36})\left(\frac{1}{4}\right)\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}\right\}=2^{34}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}$	Correct answer **Note:** $k=34$, $a=\sqrt{3}$, $b=-\sqrt{3}$	A1
			(2)
			9

	Question 7 Notes	
7. (a)	**Note**	Proof must contain the final steps of $= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ **and** $= 8\mathbf{I}$ **or** $= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$ **and** $=$ RHS
	Note	Other acceptable proofs for M1 dM1 A1 include • $\mathbf{P}^3 = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}^3$ $= \begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8\mathbf{I}$ * • $\mathbf{P}^3 = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}^3$ $= \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8\mathbf{I}$ * • $\mathbf{P}^3 = \begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8\mathbf{I}$ * • $\mathbf{P}^3 = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8\mathbf{I}$ *
(b)	**Note**	"original point" is not acceptable in place of the word "origin".
	Note	"expand" is 1st M0
	Note	"enlarge x by 2 and no change in y" is 1st M0 1st A0
	Note	Writing "120 degrees" by itself implies by convention "120 degrees anti-clockwise". So • "Rotation 120 degrees about O" is 2nd M1 2nd A1 • "Rotation 120 degrees clockwise about O" is 2nd M1 2nd A0
	Note	Writing down "centre $(0, 0)$ with scale factor 2" with no reference to "enlargement" or "enlarge" or "dilation" is 1st M0 1st A0
	Note	Writing down "120 degrees anti-clockwise about O" with no reference to "rotation" or "turn" is 2nd M0 2nd A0
	Note	Give 1st M1 1st A0 for writing "stretch parallel to x-axis and y-axis"
	Note	Give 1st M1 1st A0 for writing "stretch scale factor 2 parallel to x-axis and stretch scale factor 2 parallel to y-axis {with centre $(0, 0)$ }"
	Note	If a candidate would score M1 A1 M1 A1 in part (b) and there is an error in their solution (e.g. a third transformation given) then give M1 A1 M1 A0
(c)	**Note**	$8^{11} = 2^{33} = 8589934592$
	Note	$8^{12} = 2^{36} = 68719476736$
	Note	(their $\mathbf{P}^2$) must be a genuine attempt at $\mathbf{P}^2$ or must be for (their $\mathbf{P}^2$) seen in part (a)
	Note	Allow M1 A1 for writing $\mathbf{P}^{35} = 2^{34}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$
	Note	Stating $k = 34$, $a = \sqrt{3}$, $b = -\sqrt{3}$ from no working is M1 A1
	Note	Give M0 A0 for $\mathbf{P}^4 = 2^3\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \Rightarrow \mathbf{P}^{35} = 2^{34}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

		Question 7 Notes Continued
7. (c)	**Note**	Writing down $(8\mathbf{I})^{11}\times\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$ or $(2\mathbf{I})^{33}\times\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$ or $(8\mathbf{I})^{11}\times\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}^2$ or $(2\mathbf{I})^{33}\times\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}^2$ **with no attempt** to evaluate $\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$ is M0
	Note	Allow M1 for **applying** $\mathbf{P}^{35}=(\mathbf{P}^3)^{11}\times\mathbf{P}^2$ or $\mathbf{P}^{35}=\mathbf{P}^{33}\times\mathbf{P}^2$ E.g. Allow M1 for $\begin{pmatrix}8 & 0\\ 0 & 8\end{pmatrix}^{11}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$ or $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}^{33}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$ or $\begin{pmatrix}8^{11} & 0\\ 0 & 8^{11}\end{pmatrix}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$ or $\begin{pmatrix}2^{33} & 0\\ 0 & 2^{33}\end{pmatrix}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$ or $(8)^{11}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$ or $(2)^{33}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-2 & 2\sqrt{3}\\ -2\sqrt{3} & -2\end{pmatrix}$
	Note	Allow M1 for $(2)^{35}\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}$ **or** $(2)^{35}\begin{pmatrix}\cos 4200 & -\sin 4200\\ \sin 4200 & \cos 4200\end{pmatrix}$ **or** $(2)^{35}\begin{pmatrix}-0.5 & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -0.5\end{pmatrix}$ **or** equivalent in radians
	Note	Give M0 for $\mathbf{P}^{35}=(\mathbf{P}^3)^{11}\times\mathbf{P}^2$ by itself
	Note	Give M0 for $\mathbf{P}^{35}=\mathbf{P}^{33}\times\mathbf{P}^2$ by itself

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8. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^n = \begin{pmatrix} 1+4n & -8n \\ 2n & 1-4n \end{pmatrix}$$

(5)

(ii) A sequence of positive numbers is defined by

$$u_1 = 8, \quad u_2 = 40$$
$$u_{n+2} = 8u_{n+1} - 12u_n \qquad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 6^n + 2^n$$

(5)

Question Number	Scheme	Notes	Marks
8.	(i) $\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}^n = \begin{pmatrix}1+4n & -8n\\2n & 1-4n\end{pmatrix}$	(ii) $u_1 = 8,\ u_2 = 40,\ u_{n+2} = 8u_{n+1} - 12u_n \Rightarrow u_n = 6^n + 2^n$	
(i)	$n=1$, LHS $= \begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$, RHS $= \begin{pmatrix}1+4(1) & -8(1)\\2(1) & 1-4(1)\end{pmatrix} = \begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$	Shows or states that **either** LHS = RHS = $\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$ **or** LHS = $\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$, RHS = $\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$	B1
	(Assume the result is true for $\boldsymbol{n=k}$)		
	$\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}^{k+1} = \begin{pmatrix}1+4k & -8k\\2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$ **or** $= \begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}\begin{pmatrix}1+4k & -8k\\2k & 1-4k\end{pmatrix}$	States intention to multiply $\begin{pmatrix}1+4k & -8k\\2k & 1-4k\end{pmatrix}$ by $\begin{pmatrix}5 & -8\\2 & -3\end{pmatrix}$ **(either way round)**	M1
	$= \begin{pmatrix}5+20k-16k & -8-32k+24k\\10k+2-8k & -16k-3+12k\end{pmatrix}$ **or** $= \begin{pmatrix}5+20k-16k & -40k-8+32k\\2+8k-6k & -16k-3+12k\end{pmatrix}$ **or** $= \begin{pmatrix}5+4k & -8-8k\\2+2k & -4k-3\end{pmatrix}$	**dependent on the previous M mark** Multiplies out to give a correct un-simplified matrix with at least 3 correct elements	dM1
	$= \begin{pmatrix}1+4(k+1) & -8(k+1)\\2(k+1) & 1-4(k+1)\end{pmatrix}$	***Uses algebra*** to achieve this result with no errors	A1
	If the result is <u>true for $n=k$,</u> then it is <u>true for $n=k+1$.</u> As the result has been shown to be <u>true for $n=1$,</u> then the result <u>is true for all n</u> $(\in \mathbb{Z}^+)$		A1 **cso**
			(5)
(ii)	$\{n=1,\}$ $u_1 = 6^1 + 2^1 = 8$; $\{n=2,\}$ $u_2 = 6^2 + 2^2 = 40$	Shows $u_1 = 8$ by writing an intermediate step of e.g. $6^1 + 2^1$ or $6+2$ **and** shows $u_2 = 40$ by writing an intermediate step of e.g. $6^2 + 2^2$ or $36+4$	B1
	(Assume the result is true for $\boldsymbol{n=k}$ and $n=k+1$)		
	$\{u_{k+2} = 8u_{k+1} - 12u_k \Rightarrow\}$ $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k)$	Finds u_{k+2} by attempting to substitute $u_{k+1} = 6^{k+1} + 2^{k+1}$ and $u_k = 6^k + 2^k$ into $u_{k+2} = 8u_{k+1} - 12u_k$ Condone one slip	M1
	either $\{u_{k+2}\} = 48(6^k) + 16(2^k) - 12(6^k + 2^k)$ $= 36(6^k) + 4(2^k)$ $= 6^2(6^k) + 2^2(2^k)$ **or** $\{u_{k+2}\} = 8(6^{k+1} + 2^{k+1}) - 2(6^{k+1}) - 6(2^{k+1})$ $= 6(6^{k+1}) + 2(2^{k+1})$ **or** $\{u_{k+2}\} = 8(6^{k+1}) - 2(6^{k+1}) + 4(2^{k+2}) - 3(2^{k+2})$ **or** $\{u_{k+2}\} = 48(6^k) - 12(6^k) + 4(2^{k+2}) - 3(2^{k+2})$	Expresses u_{k+2} correctly in terms of **only** 6^k **and** 2^k **or only** 6^{k+1} **and** 2^{k+1} **or** as $8(6^{k+1}) - 2(6^{k+1}) + 4(2^{k+2}) - 3(2^{k+2})$ **or** as $48(6^k) - 12(6^k) + 4(2^{k+2}) - 3(2^{k+2})$	A1 (M1 on ePEN)
	$= 6^{k+2} + 2^{k+2}$	**dependent on the previous A mark** ***Uses algebra*** in a complete method to achieve this result with no errors	A1
	If the result is <u>true for $n=k$ and for $n=k+1$,</u> then it is <u>true for $n=k+2$.</u> As the result has been shown to be <u>true for $n=1$ and $n=2$,</u> then the result <u>is true for all n</u> $(\in \mathbb{Z}^+)$		A1 **cso**
			(5)
			10

	Question 8 Notes	
8. (i)	**Note**	**Final A1** is dependent on all previous marks being scored. It is gained by candidates conveying the ideas of **all** four underlined points **in part (i)** **either** at the end of their solution **or** as a narrative in their solution.
	Note	"Assume for $n=k$, $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^k = \begin{pmatrix}1+4k & -8k\\ 2k & 1-4k\end{pmatrix}$" satisfies the requirement "true for $n=k$"
	Note	"For $n \in \mathbb{Z}^+$, $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}^n = \begin{pmatrix}1+4n & -8n\\ 2n & 1-4n\end{pmatrix}$" satisfies the requirement "true for all n"
	Note	Give B0 for stating LHS = RHS by itself with no reference to LHS = RHS = $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$
	Note	Allow for B1 for stating either, $n=1$, $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix} = \begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix}$ or $\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix} = \begin{pmatrix}1+4 & -8\\ 2 & 1-4\end{pmatrix}$
	Note	E.g. $\begin{pmatrix}1+4k & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix} = \begin{pmatrix}1+4(k+1) & -8(k+1)\\ 2(k+1) & 1-4(k+1)\end{pmatrix}$ with no intermediate working is M1 dM0 A0 A0
	Note	E.g. Writing any of • $\begin{pmatrix}1+4k & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix} = \begin{pmatrix}5+20k-16k & -8-32k+24k\\ 10k+2-8k & -16k-3+12k\end{pmatrix} = \begin{pmatrix}1+4(k+1) & -8(k+1)\\ 2(k+1) & 1-4(k+1)\end{pmatrix}$ • $\begin{pmatrix}1+4k & -8k\\ 2k & 1-4k\end{pmatrix}\begin{pmatrix}5 & -8\\ 2 & -3\end{pmatrix} = \begin{pmatrix}5+4k & -8-8k\\ 2+2k & -4k-3\end{pmatrix} = \begin{pmatrix}1+4(k+1) & -8(k+1)\\ 2(k+1) & 1-4(k+1)\end{pmatrix}$ is M1 dM1 A1
(ii)	**Note**	Ignore $u_3 = 8u_2 - 12u_1 = 8(40) - 12(8) = 224$ as part of their solution to (i)
	Note	Ignore $\{n=3,\}$ $u_2 = 6^3 + 2^3 = 224$ as part of their solution to (i)
	Note	Full marks in (i) can be obtained for an equivalent proof where $n=k \rightarrow n=k-1$; i.e. $k \equiv k-1$
	Note	**Final A1** is dependent on all previous marks being scored. It is gained by candidates conveying the ideas of **all** four underlined points **in part (ii)** **either** at the end of their solution **or** as a narrative in their solution.
	Note	"Assume for $n=k$, $u_k = 6^k + 2^k$ and for $n=k+1$, $u_{k+1} = 6^{k+1} + 2^{k+1}$" satisfies the requirement "true for $n=k$ and $n=k+1$"
	Note	"For $n \in \mathbb{Z}^+$, $u_n = 6^n + 2^n$ " satisfies the requirement "true for all n"
	Note	Writing $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 6^{k+2} + 2^{k+2}$ with no intermediate working is M1 A0 A0 A0
	Note	E.g. Writing either • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 48(6^k) + 16(2^k) - 12(6^k + 2^k) = 6^{k+2} + 2^{k+2}$ • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 36(6^k) + 4(2^k) = 6^{k+2} + 2^{k+2}$ • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 6^2(6^k) + 2^2(2^k) = 6^{k+2} + 2^{k+2}$ • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 8(6^{k+1} + 2^{k+1}) - 2(6^{k+1}) - 6(2^{k+1}) = 6^{k+2} + 2^{k+2}$ • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 6(6^{k+1}) + 2(2^{k+1}) = 6^{k+2} + 2^{k+2}$ • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = 8(6^{k+1}) - 2(6^{k+1}) + 4(2^{k+2}) - 3(2^{k+2}) = 6^{k+2} + 2^{k+2}$ • $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = (6)(6^{k+1}) + 4(2^{k+2}) - 3(2^{k+2}) = 6^{k+2} + 2^{k+2}$ is M1 A1 A1
	Note	Writing $u_{k+2} = 8(6^{k+1} + 2^{k+1}) - 12(6^k + 2^k) = (6)6^{k+1} + 2^{k+2} = 6^{k+2} + 2^{k+2}$ with no intermediate working is M1 A0 A0 A0

	Question 8 Notes Continued	
8. (ii)	**Note**	Full marks in (i) can be obtained for an equivalent proof where e.g. • $n = k,\ n = k+1,\ \rightarrow n = k-2,\ n = k-1$; i.e. $k \equiv k-2$
8. (i), (ii)	**Note**	Referring to n as a real number their conclusion is final A0
	Note	Referring to n as any integer in their conclusion is final A0
	Note	Condone $n \in \mathbb{Z}^*$ as part of their conclusion for the final A1

Leave blank

9. The complex numbers z_1 and z_2 are given by

$$z_1 = -1 - \mathrm{i} \quad \text{and} \quad z_2 = 3 - 4\mathrm{i}$$

(a) Find the argument of the complex number $z_1 - z_2$. Give your answer in radians to 3 decimal places.

(3)

(b) Find the complex number $\frac{z_1}{z_2}$ in the form $a + \mathrm{i}b$, where a and b are rational numbers.

(3)

(c) Find the modulus of $\frac{z_1}{z_2}$, giving your answer as a simplified surd.

(2)

(d) Find the values of the real constants p and q such that

$$\frac{p + \mathrm{i}q - 8z_1}{p - \mathrm{i}q - 8z_2} = 3\mathrm{i}$$

(5)

Question Number	Scheme	Notes	Marks
9.	$z_1 = -1 - \mathrm{i},\ z_2 = 3 - 4\mathrm{i};$ (d) $\dfrac{p + \mathrm{i}q - 8z_1}{p - \mathrm{i}q - 8z_2} = 3\mathrm{i}$		
(a)	$z_1 - z_2 = -4 + 3\mathrm{i}$	$z_1 - z_2 = -4 + 3\mathrm{i}$, seen or implied	B1
	$\{z_1 - z_2 = -4 + 3\mathrm{i} \Rightarrow\}$ $\arg(z_1 - z_2) = \pi - \tan^{-1}\left(\frac{3}{4}\right)$ $\{\arg(z_1 - z_2) = \pi - 0.6435011... \Rightarrow\}$	$z_1 - z_2 = \alpha + \beta\mathrm{i};\ \alpha < 0,\ \beta > 0$ **and** uses trigonometry to find an expression for $\arg(z_1 - z_2)$ so that $\arg(z_1 - z_2)$ is in the range $(1.58..., 3.14...)$ or $(90°, 180°)$ or $(-3.15..., -4.71...)$ or $(-180°, -270°)$	M1
	$\arg(z_1 - z_2) = 2.4980915...\ \{= 2.498\ (3\text{ dp})\}$	awrt 2.498	A1
			(3)
(b) **Way 1**	$\left\{\dfrac{z_1}{z_2} =\right\}\ \dfrac{(-1-\mathrm{i})(3+4\mathrm{i})}{(3-4\mathrm{i})(3+4\mathrm{i})}$	Multiplies numerator and denominator by the conjugate of the denominator	M1
	$= \dfrac{-3 - 4\mathrm{i} - 3\mathrm{i} + 4}{9 + 16}\ \left\{= \dfrac{1 - 7\mathrm{i}}{25}\right\}$	Numerator correct (with $\mathrm{i}^2 = -1$ applied) **or** denominator correct (with $\mathrm{i}^2 = -1$ applied)	A1
	$= \dfrac{1}{25} - \dfrac{7}{25}\mathrm{i}$ or $0.04 - 0.28\mathrm{i}$	$\dfrac{1}{25} - \dfrac{7}{25}\mathrm{i}$ or $0.04 - 0.28\mathrm{i}$	A1
			(3)
(b) **Way 2**	$\dfrac{-1-\mathrm{i}}{3-4\mathrm{i}} = a + \mathrm{i}b \Rightarrow -1 - \mathrm{i} = (a + \mathrm{i}b)(3 - 4\mathrm{i})$ $\{\text{Real} \Rightarrow\}\ -1 = 3a + 4b$ $\{\text{Imaginary} \Rightarrow\}\ -1 = -4a + 3b$ $\Rightarrow a = ...$ or $b = ...$	Sets $\dfrac{z_1}{z_2} = a + \mathrm{i}b$, multiplies both sides by z_2, attempts to equate **both** the real part **and** the imaginary part of the resulting equation **and** solves to give at least one of $a = ...$ or $b = ...$	M1
	$a = \dfrac{1}{25}$ or 0.04, $b = -\dfrac{7}{25}$ or -0.28	At least one of either a or b is correct	A1
	So, $\dfrac{z_1}{z_2} = \dfrac{1}{25} - \dfrac{7}{25}\mathrm{i}$ or $0.04 - 0.28\mathrm{i}$	$\dfrac{1}{25} - \dfrac{7}{25}\mathrm{i}$ or $0.04 - 0.28\mathrm{i}$	A1
			(3)
(c)	$\left\{\left\|\dfrac{z_1}{z_2}\right\| =\right\}\ \sqrt{\left(\dfrac{1}{25}\right)^2 + \left(\dfrac{-7}{25}\right)^2}\ \left\{\textbf{or}\ \dfrac{\|z_1\|}{\|z_2\|} =\right\}\ \dfrac{\sqrt{(-1)^2 + (-1)^2}}{\sqrt{(3)^2 + (-4)^2}}$	Finds $\left\|\dfrac{z_1}{z_2}\right\|$ by applying a full Pythagoras method	M1
	$\left\{= \dfrac{\sqrt{50}}{25}\right\} = \dfrac{\sqrt{2}}{5}$	$\dfrac{\sqrt{2}}{5}$ or $\dfrac{1}{5}\sqrt{2}$	A1 **cao**
			(2)
(d)	$p + \mathrm{i}q - 8z_1 = 3\mathrm{i}(p - \mathrm{i}q - 8z_2)$ $\Rightarrow p + \mathrm{i}q - 8(-1 - \mathrm{i}) = 3\mathrm{i}(p - \mathrm{i}q - 8(3 - 4\mathrm{i}))$	Multiplies both sides by only $(p - \mathrm{i}q - 8z_2)$, and substitutes the given values for z_1 and z_2	M1
	$\Rightarrow p + \mathrm{i}q + 8 + 8\mathrm{i} = 3p\mathrm{i} + 3q - 72\mathrm{i} - 96$ $\{\text{Real} \Rightarrow\}\ p + 8 = 3q - 96$ $\{\text{Imaginary} \Rightarrow\}\ q + 8 = 3p - 72$	**dependent on the previous M mark** attempts to equate **both** the real part **and** the imaginary part of the resulting equation	dM1
		Both correct equations which can be simplified or un-simplified	A1
	$\left\{\begin{matrix} p - 3q = -104 \\ 3p - q = 80 \end{matrix}\right. \Rightarrow \left.\begin{matrix} p - 3q = -104 \\ 9p - 3q = 240 \end{matrix}\right\}$	**dependent on the previous M mark** Obtains two equations both in terms of p and q and solves them simultaneously to give at least one of $p = ...$ or $q = ...$	ddM1
	$\Rightarrow p = 43,\ q = 49$	$p = 43,\ q = 49$	A1
			(5)
			13

		Question 9 Notes
9. (a)	**Note**	Allow M1 (implied) for awrt 2.5, truncated 2.4, awrt -3.8, truncated -3.7, awrt 143°, awrt $-217°$ or truncated $-216°$
	Note	Give B1 M1 A1 for writing $\arg(z_1 - z_2) =$ awrt 2.498 from no working.
(b)	**Note**	Give 2nd A0 for writing down $\frac{1-7\text{i}}{25}$ with no reference to $\frac{1}{25} - \frac{7}{25}\text{i}$ or $0.04 - 0.28\text{i}$
	Note	Give M1 1st A1 for writing down $\frac{1-7\text{i}}{25}$ from no working in (b)
	Note	Give M1 A1 A1 for writing down $\frac{1-7\text{i}}{25} = \frac{1}{25} - \frac{7}{25}\text{i}$ or $0.04 - 0.28\text{i}$ from no working in (b)
	Note	Give M1 A1 A1 for writing down $\frac{1}{25} - \frac{7}{25}\text{i}$ or $0.04 - 0.28\text{i}$ from no working in (b)
	Note	Give 2nd A0 for simplifying a correct $\frac{1}{25} - \frac{7}{25}\text{i}$ to give a final answer of $1 - 7\text{i}$
(c)	**Note**	M1 can be implied by awrt 0.283 or truncated 0.282
	Note	Give A0 for $\frac{\sqrt{50}}{25}$ or 0.28284… without reference to $\frac{\sqrt{2}}{5}$ or $\frac{1}{5}\sqrt{2}$
	Note	Give M0 for $\sqrt{\left(\frac{1}{25}\right)^2 + \left(\frac{-7\text{i}}{25}\right)^2}$ unless recovered by later working
	Note	Give M1 A1 for writing $\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\sqrt{2}}{5}$ from no working.