Write your name here

| Surname | Other names |
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**Pearson Edexcel**
**International**
**Advanced Level**

| Centre Number | Candidate Number |
|---|---|
| | |

# Core Mathematics C34

## Advanced

Wednesday 8 November 2017 – Morning
**Time: 2 hours 30 minutes**

Paper Reference
**WMA02/01**

**You must have:**
Mathematical Formulae and Statistical Tables (Blue)

Total Marks

**Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B). Coloured pencils and highlighter pens must not be used.
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- When a calculator is used, the answer should be given to an appropriate degree of accuracy.

## Information

- The total mark for this paper is 125.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

*Turn over* ▶

Pearson

**1.** $$\mathrm{f}(x) = x^5 + x^3 - 12x^2 - 8, \qquad x \in \mathbb{R}$$

(a) Show that the equation $\mathrm{f}(x) = 0$ can be written as

$$x = \sqrt[3]{\frac{4(3x^2 + 2)}{x^2 + 1}}$$

**(3)**

(b) Use the iterative formula

$$x_{n+1} = \sqrt[3]{\frac{4(3x_n^{\;2} + 2)}{x_n^{\;2} + 1}}$$

with $x_0 = 2$, to find $x_1$, $x_2$ and $x_3$ giving your answers to 3 decimal places.

**(3)**

The equation $\mathrm{f}(x) = 0$ has a single root, $\alpha$.

(c) By choosing a suitable interval, prove that $\alpha = 2.247$ to 3 decimal places.

**(2)**

| Question Number | Scheme | Marks |
|---|---|---|
| **1(a)** | $x^5+x^3-12x^2-8=0 \Rightarrow x^5+x^3=12x^2+8$ | M1 |
| | $x^3\left(x^2+1\right)=12x^2+8 \Rightarrow x^3=\dfrac{12x^2+8}{(x^2+1)}$ or e.g. $x^3=\dfrac{4\left(3x^2+2\right)}{(x^2+1)}$ | A1 |
| | Note that going straight from $x^5+x^3=12x^2+8$ to $x^3=\dfrac{12x^2+8}{(x^2+1)}$ is acceptable for the first 2 marks but the final mark should be withheld for not explicitly showing the factorisation of the lhs | |
| | $\Rightarrow x=\sqrt[3]{\dfrac{4\left(3x^2+2\right)}{(x^2+1)}}$ or $x=\sqrt[3]{\dfrac{4\left(2+3x^2\right)}{(x^2+1)}}$ | A1* |
| | | **(3)** |
| **(b)** | $x_1=\sqrt[3]{\dfrac{4\left(3\times 2^2+2\right)}{2^2+1}}=2.237$ | M1A1 |
| | $x_2=2.246,\quad x_3=2.247$ | A1 |
| | | **(3)** |
| **(c)** | Interval $\left[2.2465,2.2475\right]\Rightarrow \text{f}\left(2.2465\right)=...,\text{f}\left(2.2475\right)=...$ | M1 |
| | $\text{f}(2.2465)=-0.0057,\ \text{f}(2.2475)=(+)0.083$ +Reason + Conclusion | A1 |
| | | **(2)** |
| | | **(8 marks)** |
| **Alt (a)** | $x=\sqrt[3]{\dfrac{4\left(3x^2+2\right)}{(x^2+1)}}\Rightarrow x^3(x^2+1)=12x^2+8$ | M1 |
| | $x^5+x^3-12x^2-8=0$ | A1 |
| | Statement Hence $\text{f}(x)=0$ | A1* |
| | | **(3)** |

(a)

M1: Attempts to write equation in the form $x^5\pm x^3=12x^2\pm 8$ or $x^3(x^2\pm 1)=12x^2\pm 8$.

A1: Intermediate line of $x^3=\dfrac{12x^2+8}{(x^2+1)}$ seen

A1*: cso with the factorisation of the lhs seen explicitly and a statement at the start that f($x$) = 0 or $x^5+x^3-12x^2-8=0$ oe e.g. $x^3\left(x^2+1\right)-4\left(3x^2+2\right)=0$

Do not be overly concerned about the cube root encompassing the whole fraction but do not allow if it is only unambiguously the numerator that has the cube root e.g. $\Rightarrow x=\dfrac{\sqrt[3]{4\left(3x^2+2\right)}}{(x^2+1)}$

**Beware of other algebraic methods of establishing the result in (a) – if in doubt send to review.**

**Alternative for part (a):**

M1: Cubes the printed result and multiplies up

A1: Obtains the required equation with no errors

A1*: Makes a conclusion (may be minimal e.g. tick, QED, # etc.) and $x^3(x^2+1)=x^5+x^3$ seen explicitly in the working

(b)

M1: Substitutes $x_0$

= 2 into iterative equation to find $x_1$ which may be implied by $\sqrt[3]{\dfrac{4\left(3\times 2^2+2\right)}{2^2+1}}$ or awrt 2.2

A1: awrt $x_1 = 2.237$

A1: awrt $x_2 = 2.246$, $x_3 = 2.247$ (Accept commas for decimal points)

(c)
M1: Attempts to evaluate f(*x*) at both ends of a suitable interval such as [2.2465, 2.2475] with evidence of substitution at least once or one correct end (1SF or 1 figure truncated). Accept a tighter interval as long as it spans the root 2.24656. (NB *x* = 2.246564001)
A1: $\text{f}(2.2465) = -0.0057, \text{f}(2.2475) = (+)0.083$ + Reason (Eg change of sign or < 0, > 0 against the appropriate value or equivalent statement) + Conclusion (E.g. a minimum “root” or *α* = 2.247 or “suitable” or “suitable interval” or “root lies between 2.2465 and 2.2475”)
Need both values correct to 1 significant figure or truncated.

Note that candidates may use $\text{g}(x) = x - \sqrt[3]{\dfrac{4\left(3x^2+2\right)}{x^2+1}}$ with suitable values – this gives e.g. $\text{g}(2.2465) = -0.000061..., \text{g}(2.2475) = (+)0.000905...$ and is an acceptable method.

If the candidate makes an attempt to compare *x* with $\sqrt[3]{\dfrac{4\left(3x^2+2\right)}{x^2+1}}$ and constructs an argument this way and you think it may be worth some credit, please send to review.

In (c) do **not** accept attempts to repeatedly apply the iterative formula to show convergence.

Leave blank

**2.** The curve $C$ has equation

$$y^3 + x^2y - 6x = 0$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $x$ and $y$.

**(5)**

(b) Hence find the exact coordinates of the points on $C$ for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

**(6)**

| Question Number | Scheme | Marks |
|---|---|---|
| **2.(a)** | $y^3+x^2y-6x=0 \Rightarrow \underline{3y^2\frac{dy}{dx}}+\overline{\overline{x^2\frac{dy}{dx}+2xy}}-6=0$ | $\overline{\overline{\text{B1}}}$ $\underline{\text{M1}}$ A1 |
| | $\Rightarrow \frac{dy}{dx}=\frac{6-2xy}{x^2+3y^2}$ | M1A1 |
| | | **(5)** |
| **(b)** | $6-2xy=0 \Rightarrow y=\frac{3}{x}$ | M1 |
| | Substitute $y=\frac{3}{x}$ into $y^3+x^2y-6x=0 \Rightarrow \frac{27}{x^3}+\frac{3x^2}{x}-6x=0$ | **d**M1 |
| | $\Rightarrow x^4=9$ | **dd**M1A1 |
| | Points $\left(\sqrt{3},\sqrt{3}\right)\left(-\sqrt{3},-\sqrt{3}\right)$ | A1A1 |
| | | **(6)** |
| | | **(11 marks)** |
| **Alt(b)** | $6-2xy=0 \Rightarrow x=\frac{3}{y}$ | M1 |
| | Substitute $x=\frac{3}{y}$ into $y^3+x^2y-6x=0 \Rightarrow y^3+\frac{9}{y^2}y-6\times\frac{3}{y}=0$ | **d**M1 |
| | $\Rightarrow y^4=9$ | **dd**M1A1 |
| | Points $\left(\sqrt{3},\sqrt{3}\right)\left(-\sqrt{3},-\sqrt{3}\right)$ | A1A1 |
| | | **(6)** |

(a)

B1: Applies the product rule to $x^2y$ to obtain $x^2\frac{dy}{dx}+2xy$

M1: Applies the chain rule to $y^3$ to obtain $Ay^2\frac{dy}{dx}$

A1: $y^3-6x=0 \Rightarrow 3y^2\frac{dy}{dx}-6=0$. i.e. $y^3$ differentiated correctly **and** $-6x \to -6$ **and** "= 0" seen or implied.

M1: Attempts to make $\frac{dy}{dx}$ the subject. This is dependent upon them having two $\frac{dy}{dx}$ terms in their derivative. One coming from their differentiation of $x^2y$ and the other from their differentiation of $y^3$

A1: Accept $\frac{dy}{dx}=\frac{6-2xy}{x^2+3y^2}$ or equivalent.

Ignore a spurious "$\frac{dy}{dx}=$" at the start but see the note above regarding where the $\frac{dy}{dx}$'s must come from for the second method mark.

**<u>If the candidate differentiates with respect to *y*, the same scheme can be applied:</u>**

B1: $x^2y \to x^2+2xy\frac{dx}{dy}$. M1: $-6x \to A\frac{dx}{dy}$ A1: $y^3-6x=0 \Rightarrow 3y^2-6\frac{dx}{dy}=0$

M1: Attempts to make $\frac{dx}{dy}$ the subject. This is dependent upon them having two $\frac{dx}{dy}$ terms in their derivative. One coming from their differentiation of $x^2y$ and the other from their differentiation of $-6x$

A1: Accept $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6-2xy}{x^2+3y^2}$ or equivalent.

**<u>If the candidate multiplies through by d*x*:</u>**

B1: $x^2y \to x^2\mathrm{d}y+2xy\mathrm{d}x$. M1: $y^3 \to Ay^2\mathrm{d}y$ A1: $y^3-6x=0 \Rightarrow 3y^2\mathrm{d}y-6\mathrm{d}x=0$

M1: $\mathrm{d}y\left(3y^2+x^2\right)=(6-2xy)\,\mathrm{d}x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=...$ A1: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6-2xy}{x^2+3y^2}$

(b)

M1: Sets the numerator of their $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ and attempts to write $x$ in terms of $y$ or vice versa. This means that their numerator must be a function of $x$ **and** $y$.

**d**M1: Substitutes their answer to $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ into $y^3+x^2y-6x=0$ to form an equation in one variable. **Dependent on the first method mark.**

**dd**M1: Reaches an equation of the form $Ax^m=Bx^n$ OR $Cy^m=Dy^n$ or equivalent e.g. $Ax^m-Bx^n=0$ OR $Cy^m-Dy^n=0$ where $m\neq n$. **Dependent on both previous method marks.**

A1: A correct equation, either $x^4=9$ or $y^4=9$ or equivalent e.g. $x^4-9=0$ or $y^4-9=0$ (May be implied by correct coordinates below)

A1: Two correct values for $x$ or $y$ or a correct pair...likely to be $x=\sqrt{3}$, $y=\sqrt{3}$

Allow equivalent exact values for $\sqrt{3}$ for this mark e.g. $\sqrt[4]{9}$ or $\frac{3}{\sqrt[4]{9}}$ or $\sqrt[4]{\frac{27}{3}}$ or awrt 1.73

A1: All 4 values correct **and simplified** i.e. $x=\pm\sqrt{3}$, $y=\pm\sqrt{3}$. The points do not have to be explicitly given as coordinates so just look for values but if any extra points/coordinates are given, this mark can be withheld. Allow $3^{\frac{1}{2}}$ for $\sqrt{3}$.

**Note** that starting with $6-2xy=x^2+3y^2$ generally will score no marks in (b)

**Note** that working with $x^2+3y^2=0$ generally will score no marks in (b) and can be ignored if seen alongside work dealing with $6-2xy=0$ unless it yields extra spurious values – in which case the final mark can be withheld – see note above.

**3.** The number of bacteria in a liquid culture is modelled by the formula

$$N = 3500(1.035)^t, \quad t \geqslant 0$$

where $N$ is the number of bacteria $t$ hours after the start of a scientific study.

(a) State the number of bacteria at the start of the scientific study.

**(1)**

(b) Find the time taken from the start of the study for the number of bacteria to reach 10 000
Give your answer in hours and minutes, to the nearest minute.

**(4)**

(c) Use calculus to find the rate of increase in the number of bacteria when $t = 8$
Give your answer, in bacteria per hour, to the nearest whole number.

**(3)**

| Question Number | Scheme | Marks |
|---|---|---|
| **3(a)** | 3500 | B1 |
| | | **(1)** |
| **(b)** | $3500(1.035)^t > 10000 \Rightarrow (1.035)^t > \frac{20}{7}$ (awrt 2.86) | M1A1 |
| | $\Rightarrow t > \frac{\log\frac{20}{7}}{\log 1.035} = 30.516 =$ 30hrs 31 mins<br>or 30 hrs 32 mins | M1A1 |
| | | **(4)** |
| **(c)** | $\frac{\mathrm{d}N}{\mathrm{d}t} = 3500(1.035)^t \ln 1.035 \Rightarrow \left.\frac{\mathrm{d}N}{\mathrm{d}t}\right\rvert_{t=8} = 3500(1.035)^8 \ln 1.035 = \text{awrt } 159$ | B1M1A1 |
| | | **(3)** |
| | | **(8 marks)** |

(a)
B1: 3500
(b)
M1: For substituting $N$ =10000 and proceeding to $(1.035)^t ... A$ where ... is $>, \geqslant, =, <$ or $\leqslant$

A1: $(1.035)^t ... \frac{20}{7}$ where ... is $>, \geqslant, =, <$ or $\leqslant$ Accept awrt 2.86 for $\frac{20}{7}$ or equivalent e.g. $\frac{10000}{3500}, \frac{100}{35}$

M1: Proceeds correctly to find a value for $t$.

Accept expressions such as $t ... \frac{\log\frac{20}{7}}{\log 1.035}$, $t ... \frac{\ln\frac{20}{7}}{\ln 1.035}$ or $t ... \log_{1.035}\frac{20}{7}$ or awrt 30.5 as evidence

A1: 30hrs 31 mins or 30hrs 32 mins (**Not** 1831 minutes)
Attempts and Trial and Improvement should be sent to review.

(c)
B1: For $\frac{\mathrm{d}N}{\mathrm{d}t} = 3500(1.035)^t \ln 1.035$ or $\frac{\mathrm{d}N}{\mathrm{d}t} = 3500\mathrm{e}^{t\ln 1.035}\ln 1.035$ (Allow $\frac{\mathrm{d}N}{\mathrm{d}t} = N\ln 1.035$)

M1: For substituting $t$ = 8 into their $\frac{\mathrm{d}N}{\mathrm{d}t}$ which is a function of $t$ but which is **not** the original function.

A1: awrt 159 (Award as soon as a correct answer is seen and isw)

**4.** (a) Prove that

$$\frac{1-\cos 2x}{\sin 2x} \equiv \tan x, \qquad x \neq \frac{n\pi}{2}$$

**(3)**

(b) Hence solve, for $0 \leqslant \theta < 2\pi$,

$$3\sec^2\theta - 7 = \frac{1-\cos 2\theta}{\sin 2\theta}$$

Give your answers in radians to 3 decimal places, as appropriate.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

**(6)**

| Question Number | Scheme | Marks |
|---|---|---|
| **4 (a)** | $\frac{1-\cos 2x}{\sin 2x} \equiv \frac{1-\left(1-2\sin^2 x\right)}{2\sin x\cos x} = \frac{2\sin x\sin x}{2\sin x\cos x}$<br>Allow $\frac{1-\cos 2x}{\sin 2x} \equiv \frac{2\sin^2 x}{2\sin x\cos x}$ | M1A1 |
| | $= \frac{\sin x}{\cos x} = \tan x$ | A1* |
| | | **(3)** |
| | **Examples** | |
| | $\frac{1-\cos 2x}{\sin 2x} \equiv \frac{1-1+2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$ | M1A1A1 |
| | $\frac{1-\cos 2x}{\sin 2x} \equiv \frac{1-\left(1-2\sin^2 x\right)}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$ | M1A1A1 |
| | $\frac{1-\cos 2x}{\sin 2x} \equiv \frac{2\sin^2 x}{2\sin x\cos x} = \tan x$ | M1A1A0 |
| | $\frac{1-\cos 2x}{\sin 2x} \equiv \frac{\cancel{2}\sin^{\cancel{2}} x}{\cancel{2}\cancel{\sin} x\cos x} = \tan x$ | M1A1A1 |
| | | **(3)** |
| **(b)** | $3\sec^2\theta - 7 = \frac{1-\cos 2\theta}{\sin 2\theta} \Rightarrow 3\sec^2\theta - 7 = \tan\theta$ | M1 |
| | $\Rightarrow 3\left(1+\tan^2\theta\right) - 7 = \tan\theta$ | M1 |
| | $\Rightarrow 3\tan^2\theta - \tan\theta - 4 = 0$ | A1 |
| | $\Rightarrow (3\tan\theta - 4)(\tan\theta + 1) = 0$ | |
| | $\Rightarrow \tan\theta = \frac{4}{3},\ \tan\theta = -1$ | **d**M1 |
| | $\theta = 0.927,\ 4.069,\ \frac{3}{4}\pi\left(2.356\right),\ \frac{7}{4}\pi\left(5.498\right)$ | A1 A1 |
| | | **(6)** |
| | | **(9 marks)** |

(a)

M1: Score for using $\cos 2x = 1 - 2\sin^2 x$ **and** $\sin 2x = 2\sin x\cos x$

If $\cos 2x = \cos^2 x - \sin^2 x$ is used first there must be an attempt to change into just $\sin^2 x$ by using the identity $\sin^2 x + \cos^2 x = 1$. Condone missing brackets for this mark.

A1: A correct intermediate line of e.g. $\frac{a\sin x\sin x}{a\sin x\cos x}$ or $\frac{a\sin^2 x}{a\sin x\cos x}$ or $\frac{1-1+2\sin^2 x}{2\sin x\cos x}$ or $\frac{1-\left(1-2\sin^2 x\right)}{2\sin x\cos x}$

A1*: Correctly proceeds to given answer with no errors or omissions including all bracketing. There must be an intermediate line of either $\frac{\cancel{2}\cancel{\sin x}\sin x}{\cancel{2}\cancel{\sin x}\cos x}$ showing cancelling or $\frac{\sin x}{\cos x}$ or $\frac{2\sin x}{2\cos x}$ before tan $x$ is seen and if their working necessitates the appearance of the 2's in the numerator and denominator and they are not shown, this mark can be withheld. If the candidate uses $\theta$ instead of $x$, the final mark should be withheld.

(b)

M1: Uses the identity from part (a) to get an equation in just $\sec^2\theta$ or $\frac{1}{\cos^2\theta}$ and $\tan\theta$

M1: Uses the identity $\sec^2\theta = \pm 1 \pm \tan^2\theta$ to get an equation in just $\tan\theta$.

A1: A correct equation in $\tan\theta$. Look for $3\tan^2\theta - \tan\theta - 4 = 0$ or equivalent.

**d**M1: Uses a correct method to solve a 3-term quadratic in $\tan\theta$ to obtain at least one value for $\tan\theta$.
**Dependent on both previous method marks.**

A1: Any two from awrt $\theta = 0.927, 4.069, \frac{3}{4}\pi(2.356), \frac{7}{4}\pi(5.498)$

A1: All four of awrt $\theta = 0.927, 4.069, \frac{3}{4}\pi(2.356), \frac{7}{4}\pi(5.498)$

If all the angles are correct but given are given to less accuracy (but at least to 2dp) then score A1A0.
The angles do not have to appear all on the same line so award marks when the correct angles are seen.
Answers in degrees: awrt 53.130, 233.130, 135, 315. Score A1 for any 2 of these but withhold the final A mark.
Ignore extra answers outside the range but deduct the final A mark for extra answers in range in an otherwise correct solution.
**If the candidate starts again and you think the attempt is worth credit then please send to review.**

Leave blank

**5.** (i) Find

$$\int\left((3x+5)^9+\mathrm{e}^{5x}\right)\mathrm{d}x$$

**(3)**

(ii) Given that $b$ is a constant greater than 2, and

$$\int_2^b \frac{x}{x^2+5}\,\mathrm{d}x = \ln\left(\sqrt{6}\right)$$

use integration to find the value of $b$.

**(5)**

| Question Number | Scheme | Marks |
|---|---|---|
| **5(i)** | $\int\left((3x+5)^9+\mathrm{e}^{5x}\right)\mathrm{d}x=\frac{(3x+5)^{10}}{30},+\frac{\mathrm{e}^{5x}}{5}(+c)$ | M1A1, B1 |
| | | **(3)** |
| **(ii)** | $\int\frac{x}{x^2+5}\,\mathrm{d}x=\frac{1}{2}\ln\left(x^2+5\right)$ | M1A1 |
| | $\int_2^b\frac{x}{x^2+5}\,\mathrm{d}x=\ln\left(\sqrt{6}\right)=\frac{1}{2}\ln\left\|b^2+5\right\|-\frac{1}{2}\ln\left\|2^2+5\right\|=\ln\left(\sqrt{6}\right)$ | M1 |
| | $\Rightarrow\ln\left(\frac{b^2+5}{9}\right)=\ln 6\Rightarrow b=7$ | **dd**M1, A1 |
| | | **(5)** |
| | | **(8 marks)** |

(i)

M1: For an integral of the form $C(3x+5)^{10}$ or $C(3x+5)^{9+1}$ where $C$ is a constant and no other powers of $(3x + 5)$

A1: $\frac{(3x+5)^{10}}{30}$. No need for + $c$. Allow un-simplified e.g. $\frac{\frac{1}{3}(3x+5)^{10}}{10}$.

B1: $\mathrm{e}^{5x}\to\frac{\mathrm{e}^{5x}}{5}$

Mark each integration independently i.e. there is no need to see everything all on one line.

(ii)

M1: For an answer of the form $C\ln k\left(x^2+5\right)$ where $C$ and $k$ are constants. Allow log for ln.

A1: $\frac{1}{2}\ln k\left(x^2+5\right)$ or $\ln k\left(x^2+5\right)^{\frac{1}{2}}$ or $\frac{1}{2}\ln k\left|x^2+5\right|$. Allow log for ln.

M1: Substitutes in both 2 and $b$ for $x$ correctly and subtracts either way around and sets equal to $\ln\left(\sqrt{6}\right)$.

**dd**M1: Removes logs correctly to obtain an equation in $b$. **Dependent on both previous M marks**.

A1: $b=7$ only. $b = \pm7$ scores A0 unless the -7 is rejected.

**<u>Note: May see integration by substitution in (ii)</u>**

E.g. $u=x^2+5$

M1: $\int\frac{x}{x^2+5}\,\mathrm{d}x=\int\frac{x}{u}\frac{\mathrm{d}u}{2x}=\frac{1}{2}\ln u$

For an answer of the form $C\ln k(u)$ where $C$ is a constant

Allow log for ln as above.

A1: $\frac{1}{2}\ln ku$

M1: $\left[\frac{1}{2}\ln u\right]_9^{b^2+5}=\frac{1}{2}\ln\left(b^2+5\right)-\frac{1}{2}\ln 9=\ln\sqrt{6}$

Substitutes in both 9 and $b^2$ + 5 correctly and subtracts either way around and sets equal to $\ln\left(\sqrt{6}\right)$.

**dd**M1: Removes logs correctly to obtain an equation in $b$. **Dependent on both previous M marks**.

A1: $b=7$ only. $b = \pm7$ scores A0 unless the -7 is rejected.

**6.**

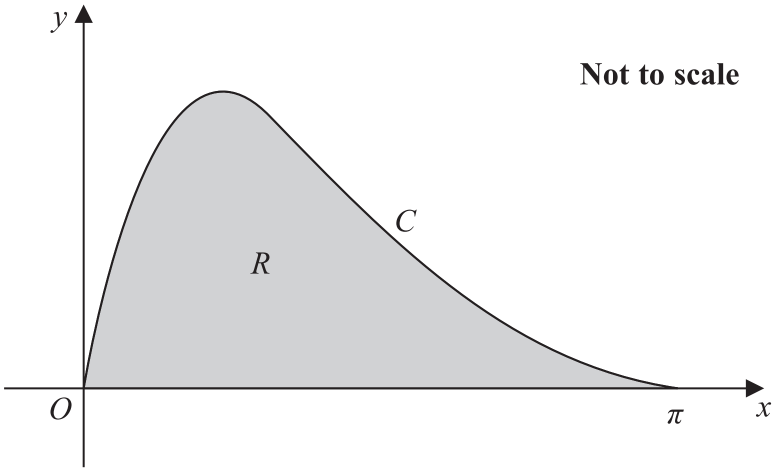


**Figure 1**

Figure 1 shows a sketch of the curve $C$ with equation $y = 2e^{-x}\sqrt{\sin x}$, $0 \leqslant x \leqslant \pi$.
The finite region $R$, shown shaded in Figure 1, is bounded by the curve and the $x$-axis.

(a) Complete the table below with the value of $y$ corresponding to $x = \frac{\pi}{2}$, giving your answer to 5 decimal places.

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
| $y$ | 0 | 0.76679 | | 0.15940 | 0 |

**(1)**

(b) Use the trapezium rule, with all the values of $y$ in the completed table, to obtain an estimate for the area of the region $R$. Give your answer to 4 decimal places.

**(3)**

(c) Given $y = 2e^{-x}\sqrt{\sin x}$, find $\frac{dy}{dx}$ for $0 < x < \pi$.

**(3)**

The curve $C$ has a maximum turning point when $x = a$.

(d) Use your answer to part (c) to find the value of $a$, giving your answer to 3 decimal places.

**(3)**

| Question Number | Scheme | Marks |
|---|---|---|
| **6 (a)** | 0.41576 | B1 |
| | | **(1)** |
| **(b)** | Strip width = $\frac{\pi}{4}$ | B1 |
| | Area $\approx \frac{1}{2}\times\frac{\pi}{4}\times\left\{0+2\left(0.76679+0.41576+0.15940\right)+0\right\}$<br>Or separate trapezia:<br>$\frac{1}{2}\times\frac{\pi}{4}\times\{0+0.766792\}+\frac{1}{2}\times\frac{\pi}{4}\times\{0.766792+0.41576\}+$<br>$\frac{1}{2}\times\frac{\pi}{4}\times\{0.41576+0.15940\}+\frac{1}{2}\times\frac{\pi}{4}\times\{0.15940+0\}$ | M1 |
| | 1.0540 | A1 |
| | | **(3)** |
| **(c)** | Uses $vu'+uv'$ : $\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{-x}\times\frac{1}{2}\left(\sin x\right)^{-\frac{1}{2}}(\cos x)-2\mathrm{e}^{-x}\left(\sin x\right)^{\frac{1}{2}}$<br>**or**<br>Uses $\frac{vu'-uv'}{v^2}$: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^{x}\times2\times\frac{1}{2}\left(\sin x\right)^{-\frac{1}{2}}(\cos x)-2\mathrm{e}^{x}\left(\sin x\right)^{\frac{1}{2}}}{\mathrm{e}^{2x}}$ | M1A1A1 |
| | | **(3)** |
| **(d)** | $\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow2\mathrm{e}^{-x}\times\frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)-2\mathrm{e}^{-x}(\sin x)^{\frac{1}{2}}=0$ | |
| | $\cos x=2\sin x$ | M1 |
| | $\tan x=\frac{1}{2}\Rightarrow x=0.464$ | dM1A1 |
| | | **(3)** |
| | | **(10 marks)** |

(a)

B1: awrt 0.41576

**(Note that degrees gives** 0.068835….and scores B0)

(b)

B1: Strip width = $\frac{\pi}{4}$ or awrt 0.785. This may be implied by seeing $\frac{1}{2}\times\frac{\pi}{4}\times\{...\}$ or $\frac{\pi}{8}\times\{...\}$ within the trapezium formula

M1: Correct structure for the trapezium formula. Do not condone missing brackets unless they are implied by subsequent work. (Allow the 0's to be omitted in the brackets)

A1: awrt 1.0540 (Not 1.054) (note that this mark is still available even if (a) is not given to the required accuracy)

**(Note that degrees gives** 0.78149…)

(c)

M1: Uses *vu*'+*uv*' with $u/v=2\mathrm{e}^{-x},u/v=\left(\sin x\right)^{0.5}$ If the rule is quoted it must be correct.

It may be implied by, for example, $u=2\mathrm{e}^{-x},v=\sqrt{\sin x}$ followed by their $u'=..,v'=..$ and $vu'+uv'$

If it is not quoted nor implied then look for an expression of the form f(*x*) ± g(*x*) where f(*x*) or g(*x*) is of the form $A\mathrm{e}^{-x}\sqrt{\sin x}$ or $A\mathrm{e}^{-x}\left(\sin x\right)^{-0.5}\cos x$ with *A* non-zero.

A1: Either term of the derivative correct

A1: Completely correct derivative $\frac{\mathrm{d}y}{\mathrm{d}x}=2\mathrm{e}^{-x}\times\frac{1}{2}\left(\sin x\right)^{-\frac{1}{2}}(\cos x)-2\mathrm{e}^{-x}\left(\sin x\right)^{\frac{1}{2}}$. Allow un-simplified and allow …+–… for … – …

Penalise poor use of powers once only e.g. $\left(\sin x\right)^{-\frac{1}{2}}$ written as $\sin x^{-\frac{1}{2}}$ unless corrected later.

**Quotient rule** on $\dfrac{2\sqrt{\sin x}}{\mathrm{e}^x}$ : M1: Uses $\dfrac{vu'-uv'}{v^2}$ with $u=2\left(\sin x\right)^{0.5}, v=\mathrm{e}^x$ If the rule is quoted it must be correct. It may be implied by, for example, $u=2\sqrt{\sin x}, v=\mathrm{e}^x$ followed by their $,u'=..,v'=..$ and $\dfrac{vu'-uv'}{v^2}$

If it is not quoted nor implied then look for an expression of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{f\left(x\right)\pm g\left(x\right)}{\left(e^x\right)^2}$ where f($x$) or g($x$) is of the form $A\mathrm{e}^x\sqrt{\sin x}$ or $A\mathrm{e}^x\left(\sin x\right)^{-0.5}\cos x$ with $A$ non-zero.

A1: Either term of the numerator correct, including the correct denominator.

A1: Completely correct derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{e}^x\times 2\times\frac{1}{2}\left(\sin x\right)^{-\frac{1}{2}}(\cos x)-2\mathrm{e}^x\left(\sin x\right)^{\frac{1}{2}}}{\mathrm{e}^{2x}}$**.** Allow un-simplified and allow …+–… for … – …

Penalise poor use of powers once only e.g. $\left(\sin x\right)^{-\frac{1}{2}}$ written as $\sin x^{-\frac{1}{2}}$ unless corrected later.

**Attempts at $y\mathrm{e}^x=2\sqrt{\sin x}$ followed by implicit differentiation should be sent to review.**

(d)

M1: Sets $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ and proceeds, using correct algebra and allowing arithmetic slips only, to $A\cos x=B\sin x$ or $A\cos x-B\sin x=0$

dM1: Divides by cos$x$ to reach tan$x$ = α where $\alpha\neq\pm1$

A1: cso awrt 0.464 (Do not allow 0.148π)

Note that in (d), some candidates may square once they reach $A\cos x=B\sin x$

E.g.

$$\cos x=2\sin x\Rightarrow\cos^2 x=4\sin^2 x\Rightarrow 5\sin^2 x=1\Rightarrow\sin x=\frac{1}{\sqrt{5}}$$

Or

$$\cos x=2\sin x\Rightarrow\cos^2 x=4\sin^2 x\Rightarrow 5\cos^2 x=4\Rightarrow\cos x=\frac{2}{\sqrt{5}}$$

In such cases, score dM1 for reaching cos $x$ = $\alpha$ or cos $x$ = $\beta$ $\alpha,\beta\neq\dfrac{1}{\sqrt{2}}$ and A1 for $x$ = awrt 0.464 but withhold the A1 if there are any extra solutions in range.

Candidates who attempt to square both sides of $\cos x-2\sin x=0$ are unlikely to progress further but if you see work that you think deserves credit, send to review.

**7.** (a) Use the binomial series to expand

$$\frac{1}{(2-3x)^3} \qquad |x| < \frac{2}{3}$$

in ascending powers of $x$, up to and including the term in $x^2$, giving each term as a simplified fraction.

**(5)**

$$\mathrm{f}(x) = \frac{4+kx}{(2-3x)^3} \qquad \text{where } k \text{ is a constant and } |x| < \frac{2}{3}$$

Given that the series expansion of $\mathrm{f}(x)$, in ascending powers of $x$, is

$$\frac{1}{2} + Ax + \frac{81}{16}x^2 + \cdots$$

where $A$ is a constant,

(b) find the value of $k$,

**(2)**

(c) find the value of $A$.

**(2)**

| Question Number | Scheme | Marks |
|---|---|---|
| **7(a)** | $2^{-3}$ or $\frac{1}{2^3}$ or $0.125$ | B1 |
| | $\frac{1}{(2-3x)^3}=(2-3x)^{-3}=\frac{1}{2^3}\left(1-\frac{3x}{2}\right)^{-3}$ | |
| | $=\frac{1}{8}\left(1+(-3)\times\left(-\frac{3x}{2}\right)+\frac{-3\times-4}{2!}\times\left(-\frac{3x}{2}\right)^2+\ldots\right)$ | M1A1 |
| | $=\frac{1}{8}+\frac{9}{16}x\,;+\frac{27}{16}x^2+\ldots$ | A1; A1 |
| | | **(5)** |
| **(b)** | $\frac{4+kx}{(2-3x)^3}=(4+kx)\left(\frac{1}{8}+\frac{9}{16}x+\frac{27}{16}x^2+\ldots\right)$ | |
| | Compares $x^2$ terms $\frac{27}{4}+\frac{9k}{16}=\frac{81}{16}\Rightarrow k=\ldots$ | M1 |
| | $k=-3$ | A1 |
| | | **(2)** |
| **(c)** | Compares $x$ terms $\frac{9}{4}+\frac{1}{8}\times{}'-3'=A\Rightarrow A=\ldots$ | M1 |
| | $A=\frac{15}{8}$ | A1 |
| | | **(2)** |
| | | **(9 marks)** |
| **7(a) ALT** | $2^{-3}$ or $\frac{1}{2^3}$ | B1 |
| | $(2-3x)^{-3}=2^{-3}+(-3)2^{-4}(-3x)+\frac{(-3)(-4)}{2}2^{-5}(-3x)^2$<br>M1: For $2^{-3}$ and the structure of at least one of the other terms correct<br>A1: Fully correct | M1A1 |
| | $=\frac{1}{8}+\frac{9}{16}x\,;+\frac{27}{16}x^2+\ldots$ | A1; A1 |
| | | **(5)** |

(a)

B1: For taking out a factor of $2^{-3}$ or $\frac{1}{2^3}$ or $\frac{1}{8}$ or $0.125$

M1: Score for the form of the binomial expansion with index $-3$

Eg $=\xcancel{\left\{\frac{1}{8}\right\}}\left[1+(-3)(**x)+\frac{(-3)(-4)}{2!}(**x)^2+\ldots\right]$ where $** \neq 1$ or $-1$

Requires 1 + … with the structure of at least one of the other terms correct as shown above.

A1: Correct un-simplified form $=\xcancel{\left\{\frac{1}{8}\right\}}\left(1+(-3)\times\left(-\frac{3x}{2}\right)+\frac{-3\times-4}{2!}\times\left(-\frac{3x}{2}\right)^2+\ldots\right)$

Condone missing brackets around the $**x$ provided they are recovered later.

A1: First two terms correct and simplified $\frac{1}{8}+\frac{9}{16}x$

A1: Third term correct and simplified $+\frac{27}{16}x^2$.

(Allow this mark from an expansion using $\frac{3x}{2}$ rather than $-\frac{3x}{2}$)

(b)

M1: Finds the sum of the coefficients of their two $x^2$ terms and sets equal to $\frac{81}{16}$ and proceeds to find a value for $k$. E.g. $4\times"\frac{27}{16}"+"\frac{9}{16}"k=\frac{81}{16}\Rightarrow k=...$

A1: cso $k=-3$ (Must come from correct work)

(c)

M1: Finds the sum of the coefficients of their two $x$ terms using their value of k and proceeds to find a value for $A$. E.g. $A=4\times"\frac{9}{16}"+"\frac{1}{8}"k$

A1: $A=\frac{15}{8}$ oe e.g. 1.875 (If $k = -3$ is obtained fortuitously in (b) allow $A=\frac{15}{8}$ here)

Leave blank

**8.** Use partial fractions, and integration, to find the exact value of $\displaystyle\int_3^4 \frac{2x^2-3}{x(x-1)}\,\mathrm{d}x$

Write your answer in the form $a + \ln b$, where $a$ is an integer and $b$ is a rational constant.
**(8)**

| Question Number | Scheme | Marks |
|---|---|---|
| **8** | $2+...$ | B1 |
| | Obtains $\frac{A}{x}+\frac{B}{x-1}$ where $A$ and $B$ are constants | M1 |
| | $\frac{3}{x}$ or $-\frac{1}{x-1}$ or $A = 3$ or $B = -1$ | A1 |
| | $\frac{3}{x}-\frac{1}{x-1}$ | A1 **(B1 on Epen)** |
| | $\int_3^4 \frac{2x^2-3}{x(x-1)}\mathrm{d}x = \int_3^4 \left(2+\frac{3}{x}-\frac{1}{x-1}\right)\mathrm{d}x$ | |
| | $=\left[2x+3\ln x-\ln(x-1)\right]_3^4$ | M1 A1ft |
| | $=\left(8+3\ln 4-\ln 3\right)-\left(6+3\ln 3-\ln 2\right)=2+\ln\left(\frac{128}{81}\right)$ | M1 A1cso |
| | | **(8 marks)** |

B1: 2 +…

M1: Obtains $\frac{A}{x}+\frac{B}{x-1}$ where $A$ and $B$ are constants

A1: $\frac{3}{x}$ or $-\frac{1}{x-1}$ or one correct constant

B1: $\frac{3}{x}-\frac{1}{x-1}$

M1: For $\int \frac{*}{x}+\frac{*}{x-1}\mathrm{d}x \to p\ln mx+q\ln n(x-1)$ where *, $p$, $q$, $m$ and $n$ are constants.

A1ft: $2x+3\ln x-\ln(x-1)$. Follow through their "2", $A$ and $B$ so look for "2"$x$+Aln$x$+Bln($x$ – 1). This mark can be withheld if the brackets are missing unless subsequent work suggests their intended presence.
M1: For substituting in 3 and 4, subtracting either way around and using correct addition or subtraction log laws at least once.

A1: cso $2+\ln\left(\frac{128}{81}\right)$ or $2+\ln\left(1\frac{47}{81}\right)$ (Do not allow $2+\ln\left(\frac{2^7}{3^4}\right)$) $2+\ln\left(\frac{128}{81}\right)+c$ is also A0

Leave blank

**9.**

$$f(x) = 2\ln(x) - 4, \qquad x > 0, \qquad x \in \mathbb{R}$$

(a) Sketch, on separate diagrams, the curve with equation

(i) $y = f(x)$

(ii) $y = |f(x)|$

On each diagram, show the coordinates of each point at which the curve meets or cuts the axes.

On each diagram state the equation of the asymptote.

**(5)**

(b) Find the exact solutions of the equation $|f(x)| = 4$

**(4)**

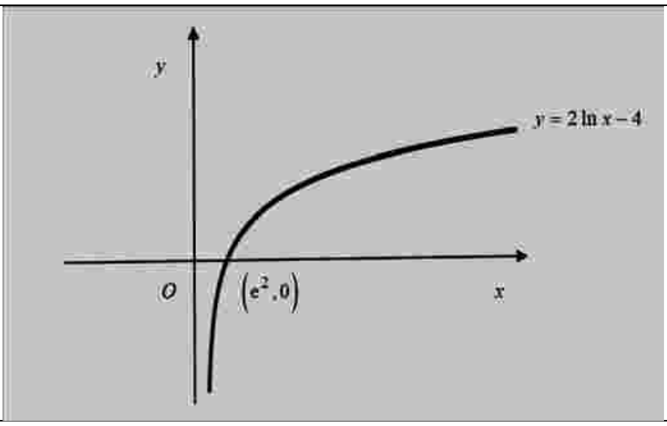
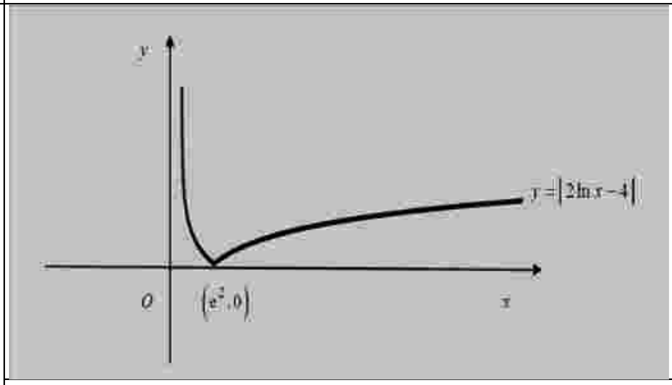
$$g(x) = e^{x+5} - 2, \qquad x \in \mathbb{R}$$

(c) Find gf($x$), giving your answer in its simplest form.

**(3)**

(d) Hence, or otherwise, state the range of gf.

**(1)**

| Question Number | Scheme | | Marks |
|---|---|---|---|
| **9(a)(i)** |  | Shape | B1 |
| | | $\left(e^2, 0\right)$ | B1 |
| | | Asymptote<br>$x = 0$ | B1 |
| | | | **(3)** |
| **(a)(ii)** |  | Shape | B1ft |
| | | Asymptote and coordinate | B1ft |
| | | | **(2)** |
| **(b)** | $2\ln x - 4 = 4 \Rightarrow \ln x = 4 \Rightarrow x = e^4$ | | M1A1 |
| | $2\ln x - 4 = -4 \Rightarrow \ln x = 0 \Rightarrow x = 1$ | | M1A1 |
| | | | **(4)** |
| **(c)** | $\mathrm{gf}(x) = e^{2\ln x - 4 + 5} - 2, = e^1 \times e^{2\ln x} - 2 = e\,x^2 - 2$ | | M1,dM1A1 |
| | | | **(3)** |
| **(d)** | $\mathrm{gf}(x) > -2$ | | B1 |
| | | | **(1)** |
| | | | **(13 marks)** |

(a)(i)

B1: For a logarithmic shaped curve in any position. For this mark be tolerant on slips of the pen at either end. See Practice and Qualification for examples.

B1: Intersection with the *x* axis at $\left(e^2, 0\right)$.

Allow $e^2$ marked on the *x* axis. Condone $\left(0, e^2\right)$ being marked on the positive *x* axis.

Do not allow $e^2$ appearing as 7.39 for this mark unless $e^2$ is seen in the body of the script.
Allow if the coordinate is given in body of script. If they are given in the body of the script and differently on the curve (save for the decimal equivalent) then the ones on the curve take precedence.

B1: **Equation** of asymptote is $x = 0$ (do not allow " *y*-axis"). Note that the curve must appear to have an asymptote at $x = 0$

(a)(ii)

B1ft: For either the correct shape or a reflection of their "negative" curve in (a) in the *x*-axis. For this to be scored it must have appeared both above and below the *x*-axis. The curve to the lhs of the intercept must appear to have the correct curvature

B1ft: Score for the correct coordinates and asymptote. Alternatively follow through on the coordinates and asymptote given in part (a) as long as the curve appeared both above and below the *x*-axis and the curve approaches the same asymptote stated in (a)(i). Do not penalise "*y*-axis" given as the asymptote twice – i.e. penalise in (a)(i) only.

**If the curves are sketched on the same axes – it must be clear which curve is which – if in doubt use review.**

(b)

M1: Sets 2ln$x$ – 4 = 4 and proceeds to $x$ = $e^{\cdots}$. This may be implied by an answer of awrt 55

A1: $x = e^4$

A correct answer only of $x = e^4$ implies both marks.

M1: Sets –2ln$x$ + 4 = 4 and proceeds to $x$ = $e^{\cdots}$ or sets 2ln$x$ + 4 = – 4 and proceeds to $x$ = $e^{\cdots}$

May be implied by an answer of e°

A1: $x$ = 1

Note that $x$ = 1 may be found by symmetry if (1, -4) is identified as a point on the original curve.

Allow M1A1 if $x$ = 1 is found by this approach.

**Alternative by squaring:**

M1: $(2\ln x - 4)^2 = 16 \Rightarrow 4(\ln x)^2 - 16\ln x + 16 = 16 \Rightarrow \ln x = ...$

Squares both sides including expanding lhs and proceeds to solve for ln$x$

M1: Proceeds from ln$x$ = … to find at least one value for $x$

A1: $x = e^4$

A1: $x$ = 1

(c)

M1: Attempts $\mathrm{gf}(x)$ the correct way around. Evidence is $\mathrm{gf}(x) = e^{2\ln x - 4 + 5} - 2$

Look for $\mathrm{gf}(x) = e^{2\ln x \pm ..}....$

dM1: Correct processing leading to an expression of the form $e^k x^2 - 2,\ k \neq 0$

(Only allow slips on the "– 4 + 5")

A1: cso $ex^2 - 2$ (Allow $e^1x^2 - 2$)

(d)

B1: Acceptable answers are: " $> -2$", $\mathrm{gf}(x) > -2$, range $> -2$, $y > -2$, $-2 < \mathrm{gf}(x) < \infty$, $(-2, \infty)$ but not $x > -2$

Allow in words e.g. gf is greater than – 2 or $y$ is bigger than –2 etc.

**10.**

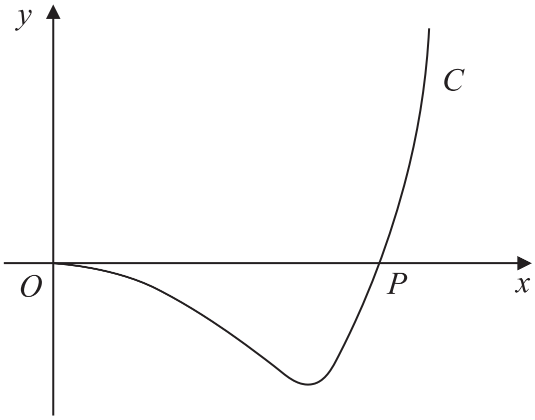


**Figure 2**

Figure 2 shows a sketch of part of the curve $C$ with parametric equations

$$x = \frac{20t}{2t + 1} \qquad y = t(t - 4), \quad t > 0$$

The curve cuts the $x$-axis at the point $P$.

(a) Find the $x$ coordinate of $P$.

**(2)**

(b) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(t - A)(2t + 1)^2}{B}$ where $A$ and $B$ are constants to be found.

**(5)**

(c) (i) Make $t$ the subject of the formula

$$x = \frac{20t}{2t + 1}$$

(ii) Hence find a cartesian equation of the curve $C$. Write your answer in the form

$$y = \mathrm{f}(x), \qquad 0 < x < k$$

where $\mathrm{f}(x)$ is a single fraction and $k$ is a constant to be found.

**(6)**

| Question Number | Scheme | Marks |
|---|---|---|
| **10 (a)** | $t(t-4)=0 \Rightarrow t=4$ Hence $x=\dfrac{20\times 4}{2\times 4+1}=\dfrac{80}{9}$ | M1A1 |
| | | **(2)** |
| **(b)** | $x=\dfrac{20t}{2t+1} \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{20(2t+1)-20t\times 2}{(2t+1)^2}=\left(\dfrac{20}{(2t+1)^2}\right)$ | M1A1 |
| | $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\dfrac{(2t-4)}{20/(2t+1)^2},=\dfrac{(t-2)(2t+1)^2}{10}$ | M1A1,A1 |
| | | **(5)** |
| | **Mark c(i) and (ii) together:** | |
| **(c)(i)** | $x=\dfrac{20t}{2t+1} \Rightarrow 2tx+x=20t \Rightarrow t(20-2x)=x \Rightarrow t=\dfrac{x}{20-2x}$ or $\dfrac{-x}{2x-20}$ | M1A1 |
| | | **(2)** |
| **(ii)** | Sub $t=\dfrac{x}{20-2x}$ into $y=t(t-4) \Rightarrow y=\dfrac{x}{20-2x}\left(\dfrac{x}{20-2x}-4\right)$ | M1 |
| | $\Rightarrow y=\dfrac{x}{20-2x}\times\left(\dfrac{x}{20-2x}-\dfrac{4(20-2x)}{20-2x}\right)$ | **d**M1 |
| | $\Rightarrow y=\dfrac{x}{20-2x}\times\left(\dfrac{9x-80}{20-2x}\right)$ | |
| | $\Rightarrow y=\dfrac{x(9x-80)}{(20-2x)^2}$, oe $\quad 0<x<10$ or $k=10$ | A1, B1 |
| | | **(4)** |
| | | **(13 marks)** |

(a)

M1: Attempts to find $x$ when $t=4$

A1: $\frac{80}{9}$ (Not 8.88… but isw if $\frac{80}{9}$ is seen)

(Ignore any attempts to find $x$ when $t=0$)

(b)

M1: Attempts to apply the quotient rule on $\dfrac{20t}{2t+1}$ with $u=20t, v=2t+1$

Alternatively applies the product rule on $20t(2t+1)^{-1}$ OR writes $\dfrac{20t}{2t+1}$ as $A-\dfrac{B}{2t+1}$ and uses the chain rule

A1: $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{20(2t+1)-20t\times 2}{(2t+1)^2}$ or $\dfrac{\mathrm{d}x}{\mathrm{d}t}=20(2t+1)^{-1}+20t\times -2(2t+1)^{-2}$

M1: Attempts to use $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ . Need to see correct chain rule seen or implied with their $\frac{dx}{dt}$ and their $\frac{dy}{dt}$ which is not $y$.

A1: $\frac{dy}{dx}$ correct and un-simplified. Score for $\frac{dy}{dx}=\frac{\frac{(2t-4)}{20}}{(2t+1)^2}$

Allow "invisible" brackets to be recovered if a correct answer appears later.

A1: $\frac{dy}{dx}=\frac{(t-2)(2t+1)^2}{10}$

(c)(i)

M1: A full attempt to make $t$ the subject of $x=\frac{20t}{2t+1}$. Need to see a correct method here with sign slips only so must reach $t=\frac{\pm x}{\pm 20\pm 2x}$.

A1: $t=\frac{x}{20-2x}$ or equivalent e.g. $t=\frac{-x}{2x-20}$

(c)(ii)

M1: Substitutes THEIR $t=\frac{x}{20-2x}$ into $y = t(t-4)$ to find $y$ in terms of $x$

dM1: Uses a correct common denominator, adapting the numerator of the second fraction. Condone sign errors only when combining **their** fractions either inside the brackets or once the brackets have been expanded. **Dependent on the first method mark.**

A1: $y=\frac{x(9x-80)}{(20-2x)^2}$ Accept exact alternatives such as $y=\frac{x(9x-80)}{4(10-x)^2}$, $y=\frac{9x^2-80x}{4(10-x)^2}$, $y=\frac{9x^2-80x}{400-80x+4x^2}$

**Note that it is possible to find y from integrating d$y$/d$x$ as a function of $x$ – send such cases to review.**

B1: Accept either the domain is $0<x<10$ or $k=10$

**11.** (a) Given $0 \leqslant h < 25$, use the substitution $u = 5 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{5 - \sqrt{h}} = -10\ln\left(5 - \sqrt{h}\right) - 2\sqrt{h} + k$$

where $k$ is a constant.

**(6)**

A team of scientists is studying a species of tree.

The rate of change in height of a tree of this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.2}\left(5 - \sqrt{h}\right)}{5}$$

where $h$ is the height of the tree in metres and $t$ is the time in years after the tree is planted.

One of these trees is 2 metres high when it is planted.

(b) Use integration to calculate the time it would take for this tree to reach a height of 15 metres, giving your answer to one decimal place.

**(7)**

(c) Hence calculate the rate of change in height of this tree when its height is 15 metres. Write your answer in centimetres per year to the nearest centimetre.

**(1)**

| Question Number | Scheme | Marks |
|---|---|---|
| **11 (a)** | $\frac{\mathrm{d}u}{\mathrm{d}h}=-\frac{1}{2}h^{-\frac{1}{2}}$ OR $\frac{\mathrm{d}h}{\mathrm{d}u}=-2(5-u)$ or $\mathrm{d}u=-\frac{1}{2}h^{-\frac{1}{2}}\mathrm{d}h$ etc. | B1 |
| | $\int\frac{\mathrm{d}h}{5-\sqrt{h}}=\int\frac{-2(5-u)\mathrm{d}u}{u}=\int\left(\frac{-10}{u}+2\right)\mathrm{d}u$ | M1dM1A1 |
| | $=-10\ln u+2u+c$ | |
| | $=-10\ln\left(5-\sqrt{h}\right)+2\left(5-\sqrt{h}\right)+c$ | M1 |
| | $=-10\ln\left(5-\sqrt{h}\right)-2\sqrt{h}+k$ | A1* |
| | | **(6)** |
| **(b)** | $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}\Rightarrow\int\frac{\mathrm{d}h}{\left(5-\sqrt{h}\right)}=\int\frac{t^{0.2}\,\mathrm{d}t}{5}$ | B1 |
| | $\Rightarrow-10\ln\left(5-\sqrt{h}\right)-2\sqrt{h}+k=\frac{t^{1.2}}{6}$ or equivalent | M1A1 |
| | Substitute $t=0,h=2\Rightarrow\ k=10\ln\left(5-\sqrt{2}\right)+2\sqrt{2}=\left(\text{awrt }15.6\right)$ | M1 |
| | Substitute $h=15\ \Rightarrow\frac{t^{1.2}}{6}=-10\ln\left(5-\sqrt{15}\right)-2\sqrt{15}+10\ln\left(5-\sqrt{2}\right)+2\sqrt{2}$ | **d**M1 |
| | $\Rightarrow t^{1.2}=39.94\Rightarrow t=21.6$ (or 21.7) | **d**M1A1 |
| | | **(7)** |
| **(c)** | $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{21.6^{0.2}\left(5-\sqrt{15}\right)}{5}=0.42$ = 42 (cm per year) | B1 |
| | | **(1)** |
| | | **(14 marks)** |
| | | |
| **Note this is where "5" is brought to lhs** | $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{t^{0.2}\left(5-\sqrt{h}\right)}{5}\Rightarrow\int\frac{5\,\mathrm{d}h}{\left(5-\sqrt{h}\right)}=\int t^{0.2}\,\mathrm{d}t$ | B1 |
| | $\Rightarrow-50\ln\left(5-\sqrt{h}\right)-10\sqrt{h}+'k'=\frac{t^{1.2}}{1.2}$ or equivalent | M1A1 |
| | Substitute $t=0,h=2\Rightarrow\ k=50\ln\left(5-\sqrt{2}\right)+10\sqrt{2}=\left(\text{awrt }78\right)$ | M1 |
| | Substitute $h=15$<br>$\Rightarrow\frac{t^{1.2}}{1.2}=-50\ln\left(5-\sqrt{15}\right)-10\sqrt{15}+50\ln\left(5-\sqrt{2}\right)+10\sqrt{2}$ | M1 |
| | $\Rightarrow t^{1.2}=39.94\Rightarrow t=21.6$ (or 21.7) | dM1A1 |
| | | (7) |

(a)

B1: For $\frac{\mathrm{d}u}{\mathrm{d}h}=-\frac{1}{2}h^{-\frac{1}{2}}$ OR $\frac{\mathrm{d}h}{\mathrm{d}u}=-2(5-u)$ or equivalent. For example accept versions such as $\mathrm{d}u=-\frac{1}{2}h^{-\frac{1}{2}}\mathrm{d}h$ and $\mathrm{d}h=-2\sqrt{h}\ \mathrm{d}u$

M1: Attempts to rewrite the integral in terms of *h* to an integral in terms of *u*. Expect to see both $\mathrm{d}h$ and $5-\sqrt{h}$ written in terms of *u* but $\mathrm{d}h \neq \mathrm{d}u$

dM1: Divides by '*u*' to reach a form $\int\left(\frac{A}{u}+B\right)\mathrm{d}u$ where *A* and *B* are constants. **Dependent on the first method mark**.

A1: $\int\left(\frac{-10}{u}+2\right)\mathrm{d}u$ or e.g. $2\int\left(\frac{-5}{u}+1\right)\mathrm{d}u$

M1: $\int\left(\frac{A}{u}+B\right)\mathrm{d}u \to A\ln u+Bu \to A\ln\left(5-\sqrt{h}\right)+B\left(5-\sqrt{h}\right)+C$

Reaches $A\ln\left(5-\sqrt{h}\right)+B\left(5-\sqrt{h}\right)$ **with or without** a constant of integration

A1*: CSO. There must have been a constant at the point of integration above and evidence that $2\left(5-\sqrt{h}\right)+c \to -2\sqrt{h}+k$ but do not accept $2\left(5-\sqrt{h}\right)+k \to -2\sqrt{h}+k$ unless it is accompanied by an explanation that 10 + *k* is a constant.

May see $-10\ln\left(5-\sqrt{h}\right)+2\left(5-\sqrt{h}\right)+c=-10\ln\left(5-\sqrt{h}\right)+10-2\sqrt{h}+c$ where 10 + *c* = *k* which is acceptable.

(b)

B1: Separates the variables.

Accept $\int\frac{\mathrm{d}h}{\left(5-\sqrt{h}\right)}=\int\frac{t^{0.2}\,\mathrm{d}t}{5}$ or equivalent, even without the integral signs.

M1: Attempts to integrate both sides. **Must see:**

$$\int\frac{\mathrm{d}h}{\left(5-\sqrt{h}\right)} \to A\ln\left(5-\sqrt{h}\right)+B\sqrt{h}\ (+k)\ \textbf{ and } \int t^{0.2}\,\mathrm{d}t \to Ct^{1.2}\ (\text{Allow } C = 1)$$

A1: $-10\ln\left(5-\sqrt{h}\right)-2\sqrt{h}+k=\frac{t^{1.2}}{6}$ or $-10\ln\left(5-\sqrt{h}\right)-2\sqrt{h}=\frac{t^{1.2}}{6}+c$

$-50\ln\left(5-\sqrt{h}\right)-10\sqrt{h}+'k'=\frac{t^{1.2}}{1.2}$ or $-50\ln\left(5-\sqrt{h}\right)-10\sqrt{h}=\frac{t^{1.2}}{1.2}+c$

**All correct with the constant appearing one side or the other (or both)**

**Note that some candidates think the *k* in part (a) is 10 – in these cases, provided all the work is correct, allow all the marks in (b) and (c) but a constant of integration must be found.**

M1: Substitutes *t* = 0 and *h* = 2 to find a value for their constant

**d**M1: Substitute $h=15$ in an equation for *t* involving a numerical constant. **Dependent on the previous method mark.**

**d**M1: **All previous method marks must have been scored**. It is for obtaining a value for *t* (even if the processing is poor).

A1: cso $t=21.6$ (years) or $t=21.7$ (years)

## Alternative:

B1: Separates the variables.

Accept $\int \frac{\mathrm{d}h}{\left(5-\sqrt{h}\right)} = \int \frac{t^{0.2}\,\mathrm{d}t}{5}$ or equivalent, even without the integral signs.

M1: Attempts to integrate both sides. **Must see:**

$$\int \frac{\mathrm{d}h}{\left(5-\sqrt{h}\right)} \to A\ln\left(5-\sqrt{h}\right)+B\sqrt{h}\left(+k\right) \textbf{ and } \int t^{0.2}\,\mathrm{d}t \to Ct^{1.2}$$

A1: $\left[-10\ln\left(5-\sqrt{h}\right)-2\sqrt{h}\right]_2^{15} = \left[\frac{t^{1.2}}{6}\right]_0^T$

**All correct with the correct limits attached (constants not needed but may be present)**

M1: Substitutes $h = 15$ and $h = 2$ and subtracts

**d**M1: Substitutes $t$ = "$T$" and $t = 0$ and subtracts – may be implied by just $\frac{t^{1.2}}{6}$. **Dependent on the previous method mark.**

**d**M1: **All previous method marks must have been scored**. It is for finding $t$.

A1: cso $t = 21.6$ (years) or $t = 21.7$ (years)

(c)

B1: 42 only

**12.** Relative to a fixed origin $O$, the lines $l_1$ and $l_2$ are given by the equations

$$l_1: \quad \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \qquad\qquad l_2: \quad \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$$

where $\lambda$ and $\mu$ are scalar parameters.

The lines $l_1$ and $l_2$ intersect at the point $A$.

(a) Write down the coordinates of $A$.

**(1)**

Given that the acute angle between $l_1$ and $l_2$ is $\theta$,

(b) show that $\sin\theta = k\sqrt{2}$, where $k$ is a rational number to be found.

**(5)**

The point $B$ lies on $l_1$ where $\lambda = 4$

The point $C$ lies on $l_2$ such that $AC = 2AB$.

(c) Find the exact area of triangle $ABC$.

**(3)**

(d) Find the coordinates of the two possible positions of $C$.

**(5)**

| Question Number | Scheme | Marks |
|---|---|---|
| **12(a)** | $\left(2,0,7\right)$ | B1 |
| | | **(1)** |
| **(b)** | $\begin{pmatrix}2\\-2\\1\end{pmatrix}.\begin{pmatrix}8\\4\\1\end{pmatrix}=16-8+1=3\times9\cos\theta$ | M1A1 |
| | $\cos\theta=\frac{1}{3}$ | A1 |
| | Uses $\sin^2\theta+\cos^2\theta=1\Rightarrow\sin\theta=\sqrt{1-\left(\frac{1}{3}\right)^2}=\sqrt{\frac{8}{9}}=\frac{2}{3}\sqrt{2}$ | M1A1* |
| | | **(5)** |
| **(c)** | $AB=\sqrt{8^2+8^2+4^2}$ OR $AB=4\times'3'$ | M1 |
| | Area = $'\frac{1}{2}ab\sin C'=\frac{1}{2}\times12\times24\times\frac{2}{3}\sqrt{2}=96\sqrt{2}$ | M1A1 |
| | | **(3)** |
| **(d)** | Attempts to find value of $\mu$ by $\frac{\text{length }AC}{\left\|\left(8,4,1\right)\right\|}=\frac{24}{9}$ | M1A1 |
| | Attempts $\begin{pmatrix}2\\0\\7\end{pmatrix}\pm\frac{8}{3}\begin{pmatrix}8\\4\\1\end{pmatrix}$ | dM1 |
| | $\left(\frac{70}{3},\frac{32}{3},\frac{29}{3}\right),\left(-\frac{58}{3},-\frac{32}{3},\frac{13}{3}\right)$ | A1A1 |
| | | **(5)** |
| | | **(14 marks)** |

(a)

B1: Accept $\left(2,0,7\right)$ or the vector equivalent

(b)

M1: Correct full method for the scalar product of the direction vectors or any multiple of the direction vectors.

A1: $\begin{pmatrix}2\\-2\\1\end{pmatrix}.\begin{pmatrix}8\\4\\1\end{pmatrix}=16-8+1=\sqrt{2^2+2^2+1^2}\times\sqrt{8^2+4^2+1^2}\cos\theta$

A correct numerical statement involving $\cos\theta$

May see the cosine rule e.g. $72=3^2+9^2-2\sqrt{2^2+2^2+1^2}\times\sqrt{8^2+4^2+1^2}\cos\theta$ or equivalent

A1: $\cos\theta=\frac{1}{3}$ oe (may be implied)

M1: Uses $\sin^2\theta+\cos^2\theta=1\Rightarrow\sin\theta=\sqrt{1-their\left(\frac{1}{3}\right)^2}$. Allow methods using a right angled triangle but must see correct work e.g. $\cos\theta=\frac{1}{3}\Rightarrow\sin\theta=\frac{\sqrt{3^2-1}}{3}$. Also this mark may be implied by a correct answer of

$\frac{2}{3}\sqrt{2}$. So allow e.g. $\cos\theta = \frac{1}{3} \Rightarrow \sin\theta = \sin\left(\cos^{-1}\frac{1}{3}\right) = \frac{2}{3}\sqrt{2}$ or

$\cos\theta = \frac{1}{3} \Rightarrow \theta = 70.52...\sin\theta = \sin(70.52...) = \frac{2}{3}\sqrt{2}$ or just $\cos\theta = \frac{1}{3} \Rightarrow \sin\theta = \frac{2}{3}\sqrt{2}$

A1*: $\sin\theta = \sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2}$ with no need to state the value of *k*.

(c)

M1: A correct method of finding $|AB| = |\boldsymbol{b} - \boldsymbol{a}| = \sqrt{8^2 + 8^2 + 4^2}$ or alternatively uses $|AB| = 4 \times '3'$

M1: Uses Area = $'\frac{1}{2}|\text{AB}| \times 2|\text{AB}|\sin\theta'$ with their $\sin\theta$ but not $'\frac{1}{2}|\text{AB}| \times \frac{1}{2}|\text{AB}|\sin\theta'$

A1: $96\sqrt{2}$

(d)

M1: Attempts to find value of $\mu$ by $\frac{\text{length } AC}{|(8,4,1)|}$ or e.g. $\sqrt{(8\mu)^2 + (4\mu)^2 + \mu^2} = "24"$ or

$(8\mu)^2 + (4\mu)^2 + \mu^2 = "24"^2$

**but not** $\sqrt{(8\mu)^2 + (4\mu)^2 + \mu^2} = 24^2$ i.e. both sides must be consistent.

A1: $\mu = (\pm)\frac{24}{9}$

dM1: Attempts to find at least one position for *C* by using $\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} \pm$ **their** $'\frac{8}{3}'\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$

A1: Either of $\left(\frac{70}{3}, \frac{32}{3}, \frac{29}{3}\right), \left(-\frac{58}{3}, -\frac{32}{3}, \frac{13}{3}\right)$.

Allow in vector form as $\begin{pmatrix} \frac{70}{3} \\ \frac{32}{3} \\ \frac{29}{3} \end{pmatrix}$ or $\frac{1}{3}\begin{pmatrix} 70 \\ 32 \\ 29 \end{pmatrix}$ or $\begin{pmatrix} -\frac{58}{3} \\ -\frac{32}{3} \\ \frac{13}{3} \end{pmatrix}$ or $\frac{1}{3}\begin{pmatrix} -58 \\ -32 \\ 13 \end{pmatrix}$ but not e.g. $\frac{1}{3}(70, 32, 29)$

A1: Both of $\left(\frac{70}{3}, \frac{32}{3}, \frac{29}{3}\right), \left(-\frac{58}{3}, -\frac{32}{3}, \frac{13}{3}\right)$.

Allow in vector form as $\begin{pmatrix} \frac{70}{3} \\ \frac{32}{3} \\ \frac{29}{3} \end{pmatrix}$ or $\frac{1}{3}\begin{pmatrix} 70 \\ 32 \\ 29 \end{pmatrix}$ or $\begin{pmatrix} -\frac{58}{3} \\ -\frac{32}{3} \\ \frac{13}{3} \end{pmatrix}$ or $\frac{1}{3}\begin{pmatrix} -58 \\ -32 \\ 13 \end{pmatrix}$ but not e.g. $\frac{1}{3}(70, 32, 29)$

**Note** that using $\overrightarrow{OC} = \overrightarrow{OA} \pm 2\overrightarrow{AB}$ is common and gives $(18, -16, 15), (-14, 16, -1)$ and generally scores **no** marks in (d).